18P401 (Pages: 2) Name : ....

Reg. No : ....

FOURTH SEMESTER M.Sc. DEGREE EXAMINATION, APRIL 2020

(CUCSS -PG)

(Mathematics)

CC17P MT4 E01 / CC18P MT4 E01 - COMMUTATIVE ALGEBRA

(2017 Admission)

Time: 3 Hours Maximum: 36 weightage

Part A

Answer ***all*** questions. Each question carries 1 weightage.

1. Let A be a ring and $m \neq (1)$ be an ideal of A such that every $x \in A - m$ is a unit in A. Show that A is a local ring.
2. If x is in the Jacobson radical of a ring A, show that $1 - xy$ is a unit in A for all y in A.
3. If p is a prime ideal show that $r(p^n) = p$ for all $n > 0$.
4. Let M be an A- module and a be an ideal of A such that $a \subseteq Ann(M)$. Show that M is an A/a - module.
5. Test whether $1 \otimes x = 3 \otimes x$ in the tensor product $Z \otimes (Z/2Z)$ for any $x \in Z/2Z$.
6. State Nakayama's lemma.
7. Define a primary ideal. Give an example of a primary ideal which is not a prime ideal.
8. Show by an example that the ring homomorphism $f : A \longrightarrow S^{-1}A$ defined by $f(x) = \frac{x}{1}$ need not be injective.
9. Verify whether $\frac{1}{2}$ and $\sqrt{2}$ are integral over Z.
10. If η is the nilradical of a ring A, show that $S^{-1}\eta$ is the nilradical of $S^{-1}A$.
11. Prove that the homomorphic image of an Artin ring is Artin.
12. Give an example of a ring which is Artinian but not Noetherian.
13. Prove that the nilradical of a Noetherian ring is nilpotent.
14. Define the dimension of a ring. Find the dimension of $\boldsymbol{R}$.

(14 × 1 = 14 Weightage)

Part B

Answer any ***seven*** questions. Each question carries 2 weightage.

15. Prove that F is a field if and only if the ideals of F are (0) and (1).
16. Let $p_1, p_2, ..., p_n$ be prime ideals of a ring A, and let a be an ideal contained in $\bigcup_{i=1}^{n} p_i$. Show that $a \subseteq p_i$ for some i.

17. Let M be a finitely generated A-module and $\{x_1, x_2, ..., x_n\}$ be a set of elements in M such that their images in M/mM form a basis for the vector space M/mM. Prove that M is generated by $\{x_1, x_2, ..., x_n\}$.

18. If S is a multiplicatively closed subset of the ring A, show that the prime ideals of $S^{-1}A$ are in one-to-one correspondence with the prime ideals of A which don't meet S.

19. Prove that the powers of a maximal ideal m are m- primary.

20. Let S be a multiplicatively closed subset of the ring A and q be a p-primary ideal. If $S \cap p = \phi$, show that $S^{-1}q$ is $S^{-1}p$- primary.

21. Let $A \subseteq B$ be rings, B integral over A. If S is a multiplicatively closed subset of A, show that $S^{-1}B$ is integral over $S^{-1}A$.

22. Let $0 \longrightarrow M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \longrightarrow 0$ be an exact sequence of A- modules. Prove that M is Noetherian if and only if M' and M'' are Noetherian.

23. Prove that every submodule of a Noetherian A-module M is finitely generated.

24. If the zero ideal is irreducible in a Noetherian ring A, show that it is primary .

(7 × 2 = 14 Weightage)

Part C

Answer any ***two*** questions. Each question carries 4 weightage.

25. Let $M' \xrightarrow{u} M \xrightarrow{v} M'' \longrightarrow 0$ be a sequence of A- modules and homomorphisms. Prove that this sequence is exact if and only of the sequence $0 \longrightarrow Hom(M'', N) \xrightarrow{\overline{v}} Hom(M, N) \xrightarrow{\overline{u}} Hom(M', N)$ is exact for all A-modules N.

26. Let M be an A- module, N and P submodules of M and S a multiplicatively closed subset of the ring A. Describe $S^{-1}M$, the module of fractions of M. Prove that
    (a) $S^{-1}M \cong S^{-1}A \otimes_A M$
    (b) $S^{-1}(N + P) = S^{-1}N + S^{-1}P$
    (c) $S^{-1}(N \cap P) = (S^{-1}N) \cap (S^{-1}P)$

27. If the zero ideal of a ring A is decomposable, prove that the set of all zero divisors of A is the union of prime ideals belonging to 0.

28. Prove that a ring A is Artin if and only if A is Noetherian and $dim\ A = 0$.

(2 × 4 = 8 Weightage)

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