18P402 (Pages: 2) Name……………………

Reg. No…………………

FOURTH SEMESTER M.Sc. DEGREE EXAMINATION, APRIL 2020

(CUCSS - PG)

CC17P MT4 E07 / CC18P MT4 E07 – ADVANCED FUNCTIONAL ANALYSIS

(Mathematics)

(2017 Admission onwards)

Time: Three Hours Maximum: 36 Weightage

PART A

Answer ***all*** questions. Each question carries 1 weightage.

1. Define the transpose F' of $F \in BL(X,Y)$. Show that $\|F'\| = \|F\|$
2. State Riesz representation theorem for $C([a,b])$
3. What is meant by moment sequences? Give an example.
4. True or false: If X is a normed space, then every bounded sequence in X' has a weak* convergent subsequence. Give reason to your answer.
5. Explain uniform convexity of a normed space. Is a strictly convex normed space always uniformly convex? Justify your answer.
6. Let X be a normed space and $A: X \to X$. Let $k \in K$ and Y be a proper closed subspace of X such that $(A - kI)(X) \subset Y$. Then show that there is some x in X such that $\|x\| = 1$ and $\|A(x) - A(y)\| \geq \frac{|k|}{2}, \forall y \in Y$
7. If X is a normed space and $A \in CL(X)$, then show that every eigen space of A corresponding to a nonzero eigen value of A is finite dimensional.
8. Describe dual X' of a normed space X and its norm.
9. Does weak convergence in a Hilbert space always imply convergence. Justify your answer.
10. Briefly explain the existence of adjoint of a bounded linear operator on a Hilbert space H.
11. Let H be a Hilbert space and $A \in BL(H)$. Prove that $\|A^2\| = \|A\|^2$, if A is normal.
12. Define numerical range of $A \in BL(H)$. Check whether it is a bounded set or not.
13. State finite dimensional spectral theorem for normal/ self-adjoint operators.
14. Define a Hilbert-Schmidt operator. Show that A^* is Hilbert-Schmidt operator if $A \in BL(H)$ is a Hilbert-Schmidt operator.

(14 x 1 = 14 Weightage)

PART B

Answer any ***seven*** questions. Each question carries 2 weightage.

15. Let X be a normed space. Show that if X' is separable, then so is X.
16. Let $1 \leq p \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$.For a fixed $y \in L^q$, define $f_y: L^p \to \boldsymbol{K}$ by

$f_y(x) = \int_a^b xy\, dm,\ \ x \in L^p$. Show that the map $F: L^q \to (L^p)'$ defined by $F(y) = f_y,\ \ y \in L^q,$ is a linear isometry from L^q to $(L^p)'$

17. If X is a separable normed space, then prove that every bounded sequence in X' has a weak* convergent subsequence.
18. Show that every closed subspace of a reflexive normed space is reflexive.
19. X is a normed space and $A \in CL(X)$. Prove that every nonzero spectral value of A is its eigen value.
20. Show that $\dim Z(A' - kI) = \dim Z(A - kI) < \infty$, for $0 \neq k \in \boldsymbol{K}$, if X is a normed space and $A \in CL(X)$.
21. Let H be a Hilbert space, G be a subspace of H and g be a continuous linear functional on G. Prove that there is a unique continuous linear functional f on H such that $f_{|G} = g$ and $\|f\| = \|g\|$.
22. Show that $R(A) = H$ if and only if A^* is bounded below, where H is a Hilbert space and $A \in BL(H)$.
23. If H is a Hilbert space and $A \in BL(H)$ is self-adjoint, then show that
$$\|A\| = \sup\{|\langle A(x), x\rangle| : x \in H, \|x\| \leq 1\}$$
24. If H is a Hilbert space and $A \in BL(H)$, then prove that $\sigma_e(A) \subset \sigma_a(A)$ and $\sigma(A) = \sigma_a(A) \cup \{k : \bar{k} \in \sigma_e(A^*)\}$.

(7 x 2 = 14 Weightage)

PART C

Answer any ***two*** questions. Each question carries 4 weightage.

25. Let $1 \leq p \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Show that the dual of $\boldsymbol{K}^n$ with the norm $\|\ \|_p$ is linearly isometric to K^n with the norm $\|\ \|_q$.
26. State and prove Riesz representation theorem.
27. Let $A \in BL(H)$ and $\omega(A)$ be the numerical range of A. Show that
    (a) $k \in \omega(A)$ if and only if $\bar{k} \in \omega(A^*)$
    (b) $\sigma_e(A) \subset \omega(A)$ and $\sigma(A)$ is contained in the closure of $\omega(A)$.
28. Let A be a nonzero compact self-adjoint operator on a Hilbert space H over $\boldsymbol{K}$. Show that there exist a finite or infinite sequence (s_n) of nonzero real numbers with $|s_1| \geq |s_2| \geq \cdots$ and an orthonormal set $\{u_1, u_2, \ldots\}$ in H such that $A(x) = \sum_n s_n \langle x, u_n\rangle u_n, x \in H$. Also prove that if the set $\{u_1, u_2, \ldots\}$ is infinite, then $s_n \to 0$ as $n \to \infty$.

(2 x 4 = 8 Weightage)

****