18P404 (Pages: 3) Name…………………….

Reg. No………………..

FOURTH SEMESTER M.Sc. DEGREE EXAMINATION, APRIL 2020

(Regular/Supplementary/Improvement)

(CUCSS - PG)

(Mathematics)

CC17P MT4 E14 / CC18P MT4 E14 - DIFFERENTIAL GEOMETRY

(2017 Admissions onwards)

Time: Three Hours Maximum: 36 Weightage

Part A

Answer ***all*** questions. Each question carries 1 weightage.

1. Sketch the level set at $c = 0, 1$ and -1 for the function $f(x_1,\ x_2) = {x_1}^2 - {x_2}^2$.
2. Sketch the vector field $\mathbb{X}(p) = (p, X(p))$, on $\mathbb{R}^2$ where $X(p) = (x_2,\ x_1)$.
3. Let $f: U \to \mathbb{R}$ be a smooth function and let $\alpha: I \to U$ be an integral curve of ∇f. Show that $\left(\frac{d}{dt}\right)(f \circ \alpha)(t) = \|\nabla f(\alpha(t))\|^2$ for all $t \in I$.
4. Let $f: U \to \mathbb{R}$ be a smooth function on U, wher U is open in $\mathbb{R}^n$. Then show that the graph of f is an n-surface, where graph$(f) = \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1}: x_{n+1} = f(x_1,\ \dots, x_{n+1})\}$.
5. What do you mean by an oriented n-surface S in $\mathbb{R}^{n+1}$.
6. Show that for an n-plane, the spherical image is a single point.
7. Show that if $\alpha: I \to \mathbb{R}^{n+1}$ is a parametrized curve with constant speed then $\ddot{\alpha}(t) \perp \dot{\alpha}(t)$ for all $t \in I$.
8. Prove that covariant derivative $\mathbb{X}'(t)$ of a smooth vector filed $\mathbb{X}$ is independent of the orientation.
9. Show that $\nabla_{\mathrm{v}}(\mathbb{X} \cdot \mathbb{Y}) = (\nabla_{\mathrm{v}} \mathbb{X}) \cdot \mathbb{Y} + \mathbb{X} \cdot (\nabla_{\mathrm{v}} \mathbb{Y})$.
10. Define curvature of a plane curve C at a point p.
11. Define the length of a parametrized curve $\alpha: I \to \mathbb{R}^{n+1}$. Prove that if α is of unit speed, then $l(\alpha) = l(I)$.
12. Find the length of the parameterized curve $\alpha: I \to \mathbb{R}^4$ where $\alpha(t) = (\cos t, \sin t, \cos t, \sin t)$, $\mathrm{I} = [0, 2\pi]$.
13. Describe the normal curvature of an n-surface S in $\mathbb{R}^{n+1}$ at a point $p \in S$.
14. Define a parametrized n −surface.

$(\mathbf{14 \times 1 = 14\ Weightage})$

Part B

Answer any ***seven*** questions. Each question carries 2 weightage.

15. Define level set and graph of a function $f: U \to \mathbb{R}$, $U \subset \mathbb{R}^{n+1}$. Show that the graph of any function $f: \mathbb{R}^n \to \mathbb{R}$, is a level set for some function $F: \mathbb{R}^{n+1} \to \mathbb{R}$.
16. Find the integral curve through $p = (1, 1)$ of the vector field $\mathbb{X}(p) = (p, X(p))$, where $X(x_1,\ x_2) = \left(-2x_2,\ \frac{1}{2}x_1\right)$.
17. What do you mean by a vector at a point p tangent to a level set. Show that the gradient of f at $p \in f^{-1}(c)$ is orthogonal to all vectors tangent to $f^{-1}(c)$.
18. Let S be an n-surface in $\mathbb{R}^{n+1}$, $S = f^{-1}(c)$, where $f: U \to R$ is such that $\nabla f(q) \neq 0$ for all $q \in S$. Suppose $g: U \to R$ is a smooth function and $p \in S$ is an extreme point of g on S. Prove that there exists a real number λ such that $\nabla g(p) = \lambda \nabla f(p)$.
19. Let S $\subset$ $\mathbb{R}^{n+1}$ be a connected n - surface in $\mathbb{R}^{n+1}$, then show that there exists on S exactly two smooth unit normal vector fields $\mathbb{N}_1 and\ \mathbb{N}_2$, and $\mathbb{N}_2\,(p) = -\mathbb{N}_1(p)$, for all $p \in S$.
20. Determine all the geodesics in S^2.
21. Compute ∇_vf where $f: \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x_1,\ x_2) = {x_1}^2 - {x_2}^2$, $v = (1, 1, \cos\theta, \sin\theta)$.
22. Prove that the 1-form η on $\mathbb{R}^2 - \{0\}$ defined by $\eta = -\frac{x_2}{x_1^2+x_2^2}dx_1 + \frac{x_1}{x_1^2+x_2^2}dx_2$ is not exact.
23. Let C be a connected oriented plane curve and let $\beta: I \to C$ be a unit speed global parameterization of C. Then show that β is either one to one or periodic. Show further that β is periodic if and only if C is compact.
24. Find the Gaussian curvature $K: S \to \mathbb{R}$, where S is the cone $x_1^2 + x_2^2 - x_3^2 = 0,\ x_3 > 0$.

$(\mathbf{7 \times 2 = 14\ Weightage})$

Part C

Answer any ***two*** questions. Each question carries 4 weightage.

25. Show that the Gauss map maps a compact connected n-surface S in $\mathbb{R}^{n+1}$ on to the unit sphere S^n.
26. (a) Let S be an n - surface in $\mathbb{R}^{n+1}$ $\alpha: I \to S$ be a parameterized curve in S, $t_0 \in I$ and $\boldsymbol{v} \in S_{\alpha(t_0)}$. Then prove that there exists a unique vector field V, tangent to S along α, which is parallel and has V $(t_0) = v$.

    (b) Let S be the unit n-sphere $\sum_{i=1}^{n+1} x_i^2 = 1$ oriented by outward unit normal vector field. Prove that the Weingarten map of S is multiplication by -1.

27. (a) Let $C = f^{-1}(r^2)$, where $f(x_1, x_2) = (x_1 - a)^2 + (x_2 - b)^2$, oriented by the outward normal $\frac{\nabla f}{\|\nabla f\|}$ Let $p = (a + r, b) \in C$. Find the local parameterization of C at p. Also compute the curvature of C at p.

    (b) Let C be a plane curve oriented by the unit normal vector field $\mathbb{N}$. Let $\alpha: I \to C$ be a unit speed local parameterization of C. For $t \in I$, $\mathbb{T}(t) = \dot{\alpha(t)}$, show that $\dot{\mathbb{T}} = (\kappa \circ \alpha)(\mathbb{N} \circ \alpha)$ and $(\mathbb{N} \dot{\circ} \alpha) = -(\kappa \circ \alpha)\mathbb{T}$
28. (a) Describe a parametrized torus in $\mathbb{R}^4$.

    (b) Show for a parameterized n-surface $\varphi: U \to \mathbb{R}^{n+1}$ in $\mathbb{R}^{n+1}$ and for $p \in U$, there exists an open set $U_1 \subset U$ about p such that $\varphi(U_1)$ is an n-surface in $\mathbb{R}^{n+1}$.

$(\mathbf{2 \times 4 = 8\ Weightage})$

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