21P301 (Pages: 2) Name……………………..

Reg. No…………………..

THIRD SEMESTER M.Sc. DEGREE EXAMINATION, NOVEMBER 2022

(CBCSS-PG)

(Regular/Supplementary/Improvement)

CC19P MTH3 C11 - MULTIVARIABLE CALCULUS AND GEOMETRY

(Mathematics)

(2019 Admission onwards)

Time: Three Hours Maximum: 30 Weightage

PART A

Answer ***all*** questions. Each question carries 1 weightage.

1. Show that if $A = L(\mathbb{R}^n, \mathbb{R}^m)$, then A is a uniformly continuous mapping of $\mathbb{R}^n$ into $\mathbb{R}^m$.
2. Show that if I is the identity operator on $\mathbb{R}^n$ then $\det(I) = 1$.
3. Show that any reparametrization of a regular curve is regular.
4. Show that if $f: \mathbb{R}^2 \to \mathbb{R}$ is a smooth function then its graph $\{(x, y, z) \in \mathbb{R}^3 : z = f(x, y)\}$ is a smooth surface with atlas consisting of the single surface patch $\sigma(u, v) = (u, v, f(u, v))$.
5. Let $f: S_1 \to S_2$ be a diffeomorphism. If σ_1 is an allowable surface patch on S_1, then show that $f \circ \sigma_1$ is an allowable surface patch on S_2.
6. Calculate the first fundamental form of $\sigma(u, v) = (u - v, u + v, u^2 + v^2)$.
7. Show that $\| \sigma_u \times \sigma_v \| = (EG - F^2)^{\frac{1}{2}}$.
8. Let κ, κ_n and κ_g be the curvature, normal curvature and geodesic curvature, respectively, of a unit-speed curve γ on an oriented surface S. Show that $\kappa^2 = \kappa_n^2 + \kappa_g^2$.

($8 \times 1 = 8$ Weightage)

PART B

Two questions should be answered from each unit. Each question carries 2 weightage.

UNIT I

9. Show that a linear operator A on a finite-dimensional vector space X is one-to-one if and only if it is onto.
10. Show that the set of all invertible linear operators on $\mathbb{R}^n$ is an open subset of $L(\mathbb{R}^n)$.
11. State and prove the contraction principle.

UNIT II

12. Show that any regular curve has a unit-speed reparametrization.
13. Find the curvature of the curve $\gamma(t) = \left(\frac{4}{5}\cos t, 1 - \sin t, -\frac{3}{5}\cos t\right)$

14. Show that the total signed curvature of a closed plane curve is an integer multiple of 2π.

UNIT III

15. Show that the transition maps of a smooth surface are smooth.
16. Compute the second fundamental form of $\sigma(u, v) = (\cos\ u, \sin\ u, v)$.
17. Show that the Gaussian curvature of a ruled surface is negative or zero.

($6 \times 2 = 12$ Weightage)

PART C

Answer any ***two*** questions. Each question carries 5 weightage.

18. Suppose f maps an open set $E \subset \mathbb{R}^n$ into $\mathbb{R}^m$. Show that $f \in \mathcal{C}'(E)$ if and only if the partial derivatives $D_j f_i$ exists and are continuous on E for $1 \leq i \leq m. 1 \leq j \leq n$.
19. State and prove the inverse function theorem.
20. Let $\gamma: (\alpha, \beta) \rightarrow \mathbb{R}^2$ be a unit-speed curve, let $s_0 \in (\alpha, \beta)$ and let φ_0 be such that $\dot{\gamma}(s_0) = (\cos\ \varphi_0, \sin\ \varphi_0)$. Show that there is a unique smooth function $\varphi: (\alpha, \beta) \rightarrow \mathbb{R}$ such that $\varphi(s_0) = \varphi_0$ and $\dot{\gamma}(s) = (\cos\ \varphi(s), \sin\ \varphi(s))$ for all $s \in (\alpha, \beta)$.
21. i) If $p \in S$ is an umbilic, then show that every tangent vector of S at p is a principal vector.

    ii) Let S be a (connected) surface of which every point is an umbilic. Show that S is an open subset of a plane or a sphere.

($2 \times 5 = 10$ Weightage)

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