22P202 (Pages: 2) Name: …..……………………

Reg. No: ….…………………..

SECOND SEMESTER M.Sc. DEGREE EXAMINATION, APRIL 2023

(CBCSS - PG)

(Regular/Supplementary/Improvement)

CC19P MTH2 C07 – REAL ANALYSIS – II

(Mathematics)

(2019 Admission onwards)

Time: Three Hours Maximum: 30 Weightage

PART A

Answer ***all*** questions. Each question carries 1 weightage.

1. Let A be the set of irrational numbers in [0, 1]. Prove that $m^*(A) = 1$.
2. Prove that the Cantor set has measure zero.
3. Prove that a real-valued function that is continuous on its measurable domain is measurable.
4. Give an example of an increasing sequence of Riemann integrable functions that converges to a function f which is not Riemann integrable.
5. Define Lebesgue integral of a function f over E.
6. Assume E has finite measure. Let $\{f_n\}$ be a sequence of measurable functions on E that converges pointwise a.e. on E to f and f is finite a.e. on E. Then prove that $\{f_n\} \to f$ in measure on E.
7. Let f be a Lipschitz function on [a, b]. Show that TV(f) $\leq$ c (b - a).
8. State Riesz - Fischer theorem.

(8 ×1 = 8 Weightage)

PART B

Answer any ***two*** questions from each unit. Each question carries 2 weightage.

UNIT I

9. Let E be a measurable set of finite outer measure. Prove that for each $\varepsilon > 0$, there is a finite disjoint collection of open intervals $\{I_k\}_{k=1}^{n}$ for which if $O = \cup_{k=1}^{n} I_k$, then $m^*(E \sim O) + m^*(O \sim E) < \varepsilon$.
10. Prove that any set E of real numbers with positive outer measure contains a subset that fails to be measurable.
11. State and prove Lusin's theorem.

UNIT II

12. State and prove the bounded convergence theorem.
13. Let E be of finite measure. Suppose the sequence of functions $\{f_n\}$ is uniformly integrable over E. If $\{f_n\} \to f$ pointwise a.e. on E then prove that f is integrable over E and $lim_{n\to\infty} \int_E f_n = \int_E f$.
14. State Fatou's lemma and show by an example that the inquality in Fatou's lemma may be strict.

UNIT III

15. Show that the function f is of bounded variation on the closed, bounded interval [a, b] if and only if it is the difference of two increasing functions on [a, b].
16. Prove that a function f on a closed, bounded interval [a, b] is absolutely continuous on [a, b] if and only if it is an indefinite integral over [a, b].
17. State and prove Jensen's inequality.

(6 × 2 = 12 Weightage)

PART C

Answer any ***two*** questions. Each question carries 5 weightage.

18. Prove that the set function Lebesgue measure, defined on the σ- algebra of Lebesgue measurable sets, assigns length to any interval, is translation invariant and is countable additive.
19. Let f be a bounded function on the closed, bounded interval [a, b]. Then prove that f is Riemann integrable over [a, b] if and only if the set of points in [a, b] at which f fails to be continuous has measure zero.
20. Let the function f be continuous on the closed, bounded interval [a, b]. Prove that f is absolutely continuous on [a, b] if and only if the family of divided difference functions $\{Diff_h(f)\}, 0 < h \leq 1$ is uniformly integrable over [a, b].
21. Prove that $L^p(E)$ is a normed linear space for $1 \leq p \leq \infty$.

(2 × 5 = 10 Weightage)

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