23P103 (Pages: 2) Name: ……………….………

Reg. No: ……………..…...

FIRST SEMESTER M.Sc. DEGREE EXAMINATION, NOVEMBER 2023

(CBCSS-PG)

(Regular/Supplementary/Improvement)

CC19P MTH1 C03 – REAL ANALYSIS – I

(Mathematics)

(2019 Admission onwards)

Time: Three Hours Maximum: 30 Weightage

Part A

Answer ***all*** questions. Each question carries 1 weightage.

1. Let X be a metric space, $E \subseteq X$ and p be a limit of E. Prove that there exists a sequence $\{p_n\}$ of elements in E such that $p_n \neq p$ for every n and $\{p_n\}$ converges to p.
2. Prove that if a function f has a limit at a point p, then it is unique.
3. State true or false and justify: "The function $f(x) = \begin{cases} \sin(\frac{1}{x}), & x \neq 0 \\ 0, & x = 0 \end{cases}$ has a discontinuity of second kind at $x = 0$."
4. If $a + \frac{b}{2} + \frac{c}{3} = 0$, where a, b, c are real constants, then prove that $a + b\,x + c\,x^2 = 0$ has at least one real root between 0 and 1.
5. Let f be a non negative and continuous function defined on $[a, b]$ such that $\int_a^b f(x)\,\mathrm{dx} = 0$. Prove that $f(x) = 0$ for all $x \in [a, b]$.
6. State true or false and justify: "Convergent series of continuous functions may have a discontinuous sum."
7. Prove that every uniformly convergent sequence of bounded functions is uniformly bounded.
8. Define equicontinuous family of functions and give an example.

(8 × 1 = 8 Weightage)

Part B

Answer any ***two*** questions from each unit. Each question carries 2 weightage.

UNIT I

9. Let X be a metric space and $K \subseteq Y \subseteq X$. Prove that K is compact relative to X if and only if K is compact relative to Y.
10. Let P be a perfect set in $\mathbb{R}^k$. Prove that P is uncountable.
11. Prove that continuous image of a connected set is connected.

UNIT II

12. State and prove Mean Value Theorem.
13. Prove that a continuous function defined on $[a, b]$ is Riemann-Stieltjes integrable.
14. If f is Riemann integrable over $[a, b]$ and if there is a differentiable function F such that $F'(x) = f(x)$, then prove that $\int_a^b f(x)\ dx = F(b) - F(a)$.

UNIT III

15. State and prove Cauchy criterion for uniform convergence of sequence of functions.
16. Check the uniform convergence of the series $\sum_{n=1}^{\infty}(-1)^n \frac{x^2+n}{n^2}$.
17. Prove that there exists a continuous function on the real line which is nowhere differentiable.

(6 × 2 = 12 Weightage)

Part B

Answer any ***two*** questions. Each question carries 5 weightage.

18. Let X and Y be metric spaces and $f: X \to Y$ be a continuous function.
    (a) If X is compact, then prove that f is uniformly continuous.
    (b) If X is compact and f is a bijective map, then prove that the map $f^{-1} : Y \to X$ defined by $f^{-1}(f(x)) = x$ is continuous, for $x \in X$.
    (c) Can we drop compactness of X in (b)? Justify.
19. State and prove L'Hospital's rule.
20. Let $f \in \mathcal{R}(\alpha)$ and $g \in \mathcal{R}(\alpha)$ on $[a, b]$.
    (a) If $m \leq f \leq M$, ϕ is continuous on $[m, M]$, and $h(x) = \phi(f(x))$ on $[a, b]$, then prove that $h \in \mathcal{R}(\alpha)$ on $[a, b]$.
    (b) Prove that $fg \in \mathcal{R}(\alpha)$.
    (c) Prove that $|f| \in \mathcal{R}(\alpha)$ and $\left|\int_a^b f\ dx\right| \leq \int_a^b |f| dx$.
21. If f is a continuous complex function on $[a, b]$, then prove that there exists a sequence of polynomials $\{P_n\}$ such that $\{P_n\}$ converges uniformly to f on $[a, b]$.

(2 × 5 = 10 Weightage)

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