22P302 (Pages: 2) Name: ....

Reg.No: ....

THIRD SEMESTER M.Sc. DEGREE EXAMINATION, NOVEMBER 2023

(CBCSS - PG)

(Regular/Supplementary/Improvement)

CC19P MTH3 C12 - COMPLEX ANALYSIS

(Mathematics)

(2019 Admission onwards)

Time : 3 Hours Maximum : 30 Weightage

Part A

Answer ***all*** questions. Each question carries 1 weightage.

1. Consider the stereographic projection between $\mathbb{C}_\infty$ and $S = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\}$. For each of the points 0 and $3+2i$ of $\mathbb{C}$ give the corresponding points of the unit sphere ***S***.
2. Find the fixed points of a dilation and the inversion on $\mathbb{C}_\infty$.
3. State and prove symmetry principle.
4. State and prove Cauchy's estimate.
5. Let $\gamma : [0, 2\pi] \to \mathbb{C}$ given by $\gamma(t) = e^{it}$. Evaluate $\int_\gamma \frac{1}{z} dz$.
6. Using residue theorem evaluate $\int_0^\infty \frac{1}{1+x^2} dx$.
7. For $|a| < 1$, let $\varphi_a(z) = \frac{z-a}{1-\bar{a}z}$. Prove that $\varphi_a'(a) = \frac{1}{1-|a|^2}$
8. Hadamard three cycle theorem.

(8 × 1 = 8 Weightage)

Part B

Answer any ***two*** questions each unit. Each question carries 2 weightage.

UNIT - I

9. For a given power series $\sum_{n=0}^{\infty} a_n (z-a)^n$, let $\frac{1}{R} = \limsup |a_n|^{1/n},\ 0 \le R \le \infty$. Prove the following

   (a) If $|z-a| < R$, the series converges absolutely. (b) If $|z-a| > R$, the series diverges.

10. Let $f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$ have radius of convergence $R > 0$. Prove that the function f is infinitely differentiable on $B(a;R)$ and furthermore $f^{(k)}(z)$ is given by the series $\sum_{n=k}^{\infty} n(n-1)(n-2)\dots(n-k+1)a_n(z-a)^{n-k}$ for all $k \ge 1$ and $|z-a| < R$. Also for $n \ge 0,\ a_n = \frac{1}{n!} f^{(n)}(a)$.

11. Let z_1, z_2, z_3, z_4 be four distinct points in $\mathbb{C}_\infty$. Prove that (z_1, z_2, z_3, z_4) is a real number iff all four points lie on a circle.

UNIT - II

12. Let γ be a closed rectifiable curve in $\mathbb{C}$. Prove that $n(\gamma; a)$ is constant for *a* belonging to a component of $G = \mathbb{C} - \{\gamma\}$. Also prove that $n(\gamma; a) = 0$ for *a* belong to the unbounded component of G.

13. State and prove the first version of Cauchy's integral formula.

14. Prove that if G be an open set and $f : G \to \mathbb{C}$ be a differentiable function, then *f* is analytic on G.

UNIT - III

15. If f has an isolated singularity at *a*, prove that $z = a$ is a removable singularity iff $\lim_{z \to a}(z - a)f(z) = 0$.

16. State and prove the residue theorem.

17. Let *f* be meromorphic in G with zeros $z_1, z_2, \ldots, z_n$ and poles $P_1, P_2, \ldots, P_m$ repeated according to their multiplicity. If *g* is analytic in G and γ is a closed rectifiable curve in G with $\gamma \approx 0$ and not passing through any z_i or P_j. Prove that $\frac{1}{2\pi i}\int_\gamma g\frac{f'}{f} = \sum_{i=1}^{n} g(z_i)n(\gamma; z_i) - \sum_{j=1}^{m} g(P_j)n(\gamma; P_j)$.

(6 × 2 = 12 Weightage)

Part C

Answer any ***two*** questions. Each question carries 5 weightage.

18. Let *u* and *v* be real valued functions defined on a region G and suppose that *u* and *v* have continuous partial derivatives. Prove that $f : G \to \mathbb{C}$ defined by $f(z) = u + iv$ is analytic iff *u* and *v* satisfy the Cauchy-Riemann equations.

19. Let $\gamma : [a, b] \to \mathbb{C}$ is of bounded variation and suppose that $f : [a, b] \to \mathbb{C}$ is continuous. Prove that there is a complex number I such that for every $\epsilon > 0$ there is a $\delta > 0$ such that when $P = \{t_0 < t_1 < \ldots < t_m\}$ is a partition of $[a, b]$ with $\|P\| = max\{(t_k - t_{k-1}) : 1 \le k \le m\} < \delta$ then $\left|I - \sum_{k=1}^{m} f(\tau_k)\left[\gamma(t_k) - \gamma(t_{k-1})\right]\right| < \epsilon$ for whatever choice of points $\tau_k,\ t_{k-1} \le \tau_k \le t_k$.

20. Prove that if γ_0 and γ_1 are two closed rectifiable curves in G with $\gamma_0 \sim \gamma_1$, then $\int_{\gamma_0} f = \int_{\gamma_1} f$ for every function *f* analytic on G.

21. State and prove all the three versions of maximum modulus principles.

(2 × 5 = 10 Weightage)

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