24P202 (Pages: 2) Name: ……………………….

Reg. No: ……………………..

SECOND SEMESTER M.Sc. DEGREE EXAMINATION, APRIL 2025

(CBCSS - PG)

(Regular/Supplementary/Improvement)

CC19P MTH2 C07 – REAL ANALYSIS - II

(Mathematics)

(2019 Admission onwards)

Time: Three Hours Maximum:30 Weightage

Part A

Answer ***all*** questions. Each question carries 1 weightage.

1. If A and B are two measurable sets, then prove that $m(A \cup B) + m(A \cap B) = m(A) + m(B)$.
2. Let $f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$. Is f continuous $a.e$? Justify.
3. Let E be a measurable set and $\{x \in E : f(x) = c\}$ is a measurable set for every extended real number c. Is f a measurable function? Justify.
4. Let E be a measurable set and f be a non-negative integrable function on E. Prove that f is finite valued $a.e.$ on E.
5. Let f be a non-negative bounded measurable function on a set E of finite measure. If $\int_E f = 0$, then prove that $f = 0$ $a.e.$ on E.
6. Let f be an integrable function on a measurable set E. Prove that for every $\varepsilon > 0$, there exists a $E_0 \subseteq E$ such that $m(E_0) < \infty$ and $\int_{E-E_0} |f| < \varepsilon$.
7. Let f and g be two real valued functions on (a, b). Show that $\underline{D}(f) + \underline{D}(f) \leq \underline{D}(f + g) \leq \overline{D}(f + g) \leq \overline{D}(f) + \overline{D}(g)$ on (a, b).
8. Let φ be a real valued convex function on (a, b). For any $s, t \in (a, b)$ with $s < t$, prove that $\frac{\varphi(x) - \varphi(s)}{x - s} \leq \frac{\varphi(t) - \varphi(x)}{t - x}$ for all $s < x < t$.

(8 × 1 = 8 Weightage)

Part B

Answer any ***two*** questions from each unit. Each question carries 2 weightage.

Unit - I

9. State and prove outer approximation of measurable sets.
10. If f and g are two measurable functions on a measurable set E, then prove that $\alpha f + \beta g$ and fg are measurable functions for any α and β.

11. Let f be a simple function defined on a measurable set E. Prove that, for any $\varepsilon > 0$, there exist a continuous function g on $\mathbb{R}$ and a closed set F such that $f = g$ on F and $m(E - F) < \varepsilon$.

Unit - II

12. Let $\{f_n\}$ be an increasing sequence of non-negative measurable functions on E. If $f_n \to f$ pointwise $a.e.$ on E, then prove that $\lim_{n\to\infty} \int_E f_n = \int_E f$. If $\{f_n\}$ is monotonically decreasing, does this result hold? Justify.
13. Let $\{f_n\}$ and $\{g_n\}$ be two sequences of functions which are uniformly integrable and tight over a measurable set E. Prove that the sequence $\{f_n + g_n\}$ is uniformly integrable and tight over E.
14. Let E be a measurable set of finite measure and $\{f_n\}$ converges to f point wise on E. Prove that $\{f_n\}$ converges to f in measure. Is the converse true ? Justify.

Unit - III

15. Prove that every Lipschitz function defined on a closed and bounded interval $[a, b]$ is absolutely continuous. Is the converse true? Justify.
16. State and prove Jordan's theorem.
17. State and prove Holder's inequality.

(6 × 2 = 12 Weightage)

Part B

Answer any ***two*** questions. Each question carries 5 weightage.

18. (a) Prove that outer measure of an interval is its length.

    (b) Prove that every interval is measurable.
19. Let f and g be bounded measurable functions on a set of finite measure E.

    (a) Then for any α and β, prove that $\int_E (\alpha f + \beta g) = \alpha \int_E f + \beta \int_E g$

    (b) If $f \le g$ on E, then prove that $\int_E f \le \int_E g$.
20. State and prove Lebesgue theorem for Riemann integrability.
21. Let E be a measurable set and $1 \le p \le \infty$. Prove that every rapidly Cauchy sequence in $L^p(E)$ converges both with respect to the $L^p(E)$ norm and pointwise $a.e.$ on E to a function in $L^p(E)$. Hence prove that $L^p(E)$ is a Banach space.

(2 × 5 = 10 Weightage)

****