C 83624 (Pages : 3) Name.............15...........

Reg. No..................................

SECOND SEMESTER M.Sc. DEGREE EXAMINATION, JUNE 2015

(CUCSS)

Mathematics

MT 2C 07—REAL ANALYSIS – II

Time : Three Hours Maximum : 36 Weightage

Part A

Short answer questions.
Answer **all** *questions.*
Each question has 1 weightage.

1. Let $A \in L(\mathbb{R}^n, \mathbb{R}^m)$ and $B \in L(\mathbb{R}^m, \mathbb{R}^k)$. Prove that $\|BA\| \le \|B\| \|A\|$.

2. Let X and Y be vector spaces and let $A \in L(X, Y)$ be such that for all $x \in X$ $Ax = 0$ implies $x = 0$. Prove that A is one to one.

3. Let $f : \mathbb{R}^3 \to \mathbb{R}^1$ be given by $f(x, y, z) = x^3 + y^3 + z^3 + x^2 + y^2 + z^2$. Find the gradient of f at (2, 3, 1).

4. State inverse function theorem.

5. Let $f = (f_1, f_2)$ be the mapping of $\mathbb{R}^2$ into $\mathbb{R}^2$ given by $f_1(x, y) = e^x \cos y$, $f_2(x, y) = e^x \sin y$. Show that the Jacobian of f is not zero at any point of $\mathbb{R}^2$.

6. Let $\mathcal{M}$ be a σ- algebra and let $\{E_i\}$ be a sequence of elements in $\mathcal{M}$. Prove that $\bigcap_{i=1}^{\infty} E_i \in \mathcal{M}$.

7. Prove that if $m^*(A) = 0$, then $m^*(A \cup B) = m^*(B)$.

8. Let $\{E_i\}$ be a sequence of disjoint measurable sets and A be any set. Prove that :

$$m^*\left(A \cap \bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} m^*(A \cap E_i).$$

9. Is the characteristic function $\chi_{(0,1)}$ measurable ? Justify your answer.

Turn over

10. Let f and g be measurable functions defined on a set E of finite measure. If $f = g$ *a. e.*, then p

that $\int_E f = \int_E g$.

11. Let f be a measurable function. Prove that f^+ and f^- are measurable. Also prove that $f = f^+ -$

12. Let $\{f_n\}$ be a sequence of measurable functions such that $f_n \to f$ in measure. If $f_n \to f$

Justify your answer.

13. For functions f and g, prove that $D_+ (f + g) \leq D_+ f + D_+ g$.

14. If f is absolutely continuous on $[a, b]$ and if $f(x) \neq 0$ for all $x \in [a, b]$, then prove that $\frac{1}{f}$ is absol

continuous on $[a, b]$.

(14 × 1 = 14 weigh

Part B

Answer any **seven** *from the following ten questions.*
Each question has weightage 2.

15. Let $f(x, y) = \begin{cases} 0 & \text{if } (x, y) = (0, 0) \\ \dfrac{x^3}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \end{cases}$ and u be any unit vector in $\mathbb{R}^2$. Show that the direct

derivative $(D_u f)(0, 0)$ exists.

16. Let $[A]_1$ be the matrix obtained from the matrix [A] by interchanging two columns. Prove
det $[A]_1 = -$ det [A].

17. Prove that the outer measure is translation invariant.

18. Let E be a measurable set and let $\epsilon > 0$. Prove that there is an open set $O \supset E$ such
$m^*(O \sim E) < \epsilon$.

19. Let $E_1, E_2, \ldots E_n$ be a disjoint collection of measurable sets and let $\varphi = \sum_{i=1}^{n} a_i m(E_i)$. If $m(E_i) <$

each i, then prove that $\int \varphi = \sum_{i=1}^{n} a_i m(E_i)$.

20. Let $f:[0,1]\to\mathbb{R}$ be given by $f(x)=\begin{cases}0 & \text{if } x \text{ is rational}\\ n & \text{if } x \text{ is irrational}\end{cases}$ where n is the number of zeros immediately after decimal point in the representation of x. Show that f is measurable and evaluate $\int_{[0,1]} f$.

21. Let $\{f_n\}$ be a sequence of non-negative measurable functions that converge to f and let $f_n \le f$ for each n. Prove that $\int f = \lim \int f_n$.

22. Show that if f is integrable over a measurable set E, then $\left|\int f\right| \le \int |f|$. When does equality occur ? Justify your answer.

23. If f is of bounded variation on $[a, b]$, then prove that $f'(x)$ exists for almost all x in $[a, b]$.

24. Prove that absolutely continuous functions on $[a, b]$ are of bounded variation on $[a, b]$.

(7 × 2 = 14 weightage)

Part C

Answer any **two** *from the following four questions.*
Each question has weightage 4.

25. Let $E \subset \mathbb{R}^n$ be an open set and let $f: E \to \mathbb{R}^m$ be a mapping differentiable at a point $x \in E$. Prove that the partial derivatives $(D_j f_i)(x)$ exist and $f'(x)\, e_j = \sum_{i=1}^{m} (D_j f_i)(x)\, u_i$ where $1 \le j \le n$.

26. (i) Prove that there exists a non-measurable set.
    (ii) Prove that Cantor set is of measure zero.

27. (i) Prove that for each $a \in \mathbb{R}$, the interval (a, ∞) is measurable.
    (ii) Let f and g be non-negative measurable functions defined on a measurable set E. Prove that $\int_E f + g = \int_E f + \int_E g$.

28. (i) Let $\{f_n\}$ be a sequence of measurable functions that converges in measure to f. Prove that there is a subsequence $\{f_{n_k}\}$ that converges to f almost everywhere.
    (ii) Let f be an integrable function on $[a, b]$ and let $F(x) = F(a) + \int_a^x f(t)\, dt$. Prove that $F'(x) = f(x)$ for almost all x in $[a, b]$.

(2 × 4 = 8 weightage)