D 91595 (Pages : 3) Name..............16..........

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THIRD SEMESTER M.Sc. DEGREE EXAMINATION, DECEMBER 2015

(CUCSS)

Mathematics

MT 3C 12—FUNCTIONAL ANALYSIS—I

Time : Three Hours Maximum : 36 Weightage

Part A

*Answer **all** questions.*

Each question carries 1 weightage.

1. Prove or disprove : A sequence in a metric space is bounded in X iff it is Cauchy.
2. Show that the metric space l^{∞} is complete.
3. Define n^{th} Dirichlet Kernel D_n and evaluate $\int_{-\pi}^{\pi} D_n(t)\,dt$.
4. State Riesz's lemma.
5. Let Y be a subspace of a normed space X. Show that $Y^0 \neq \phi$ iff $Y = X$.
6. Illustrate with an example that a linear map on a linear space X may be continuous with respect to some norm on X, but discontinuous with respect to another norm on X.
7. Let X be an inner product space and $x \in X$. Show that $\langle x, y \rangle = 0$ for all $y \in X$ iff $x = 0$.
8. Show that if E and F are closed subsets of a Hilbert space H and $E \perp F$, then $E + F$ is closed in H.
9. Let X be an inner product space. Show that if $E \subset X$ is convex, then there exists at most one best approximation from E to any $x \in X$.
10. Let X be a normed space over K. Let $\{a_1, a_2, \ldots, a_m\}$ be a linearly independent set in X. Show that there are $f_1, f_2, \ldots, f_m$ in X' such that $f_j(a_i) = \delta_{ij}, 1 \le i, j \le m$.

Turn over

11. With usual notations, show that $C_c(T)$ is not closed in $C_o(T)$.

12. Define Schauder basis for a normed space X and show that if there is a Schauder basis for a normed space X, then X must be separable.

13. Show that the linear space C_{oo} cannot be a Banach space in any norm.

14. State Uniform boundedness principle and interpret it geometrically.

(14 × 1 = 14 weightage)

Part B

Answer any **seven** *questions.*
Each question carries 2 weightage.

15. Show that the set of all polynomials in one variable is dense in $c([a, b])$ with the sup metric.

16. Show that the metric space $L^p([a, b])$ is separable for $1 \leq p < \infty$, but the metric space $L^\infty([a, b])$ is not separable.

17. Show that for all $x \in K^n$;

$$\|x\|_\infty \leq \|x\|_2 \leq \|x\|_1 \text{ and}$$

$$\|x\|_1 \leq \sqrt{n}\,\|x\|_2 \leq n\,\|x\|_\infty .$$

18. Let $X = K^3$ for $x = (x(1), x(2), x(3)) \in X$, let $\|x\| = \left[\left(|x(1)|^2 + |x(2)|^2\right)^{3/2} + |x(3)|^3\right]^{1/3}$. Show that $\| \ \|$ is a norm on K^3.

19. Let X and Y be normed spaces and Z be a closed subspace of X. Show that if $\tilde{F} \in BL\left(X/Z, Y\right)$ and we let $F(x) = \tilde{F}(x + z)$ for $x \in X$, then $F \in BL(X, Y)$ and $\|F\| = \|\tilde{F}\|$.

20. Show that if a non-zero Hilbert space H over K has a countable orthonormal basis then H is linearly isometric to K^n for some n, or to l^2.

21. Let E be a non-empty closed convex sub-set of a Hilbert space H. Show for each $x \in H$, there exists a unique best approximation from E to x.

22. Let $X = K^2$ with the norm $\| \ \|_\infty$. Consider $Y = \{(x(1), x(2) \in X) : x(1) = x(2)\}$, and define $g \in Y'$ by $g(x(1), x(2)) = x(1)$.

Show that the Hahn-Banach extensions of g to X are given by :

$f(x(1), x(2)) = t\, x(1) + (1-t)\, x(2)$, where $t \in [0,1]$ is fixed.

23. Show that a normed space X is a Banach space iff every absolutely summable series of elements in X is summable in X.

24. Show that a subset E of a normed space X is bounded in X iff $f(E)$ is bounded in K for every $f \in X'$

(7 × 2 = 14 weightage)

Part C

Answer any **two** *questions.*
Each question carries 4 weightage.

25. Show that every finite dimensional subspace of a normed space X is closed in X.

26. Let $\{u_\alpha\}$ be an orthonormal set in a Hilbert space H. Show that $\{u_\alpha\}$ is an orthonormal basis for H iff space $\{u_\alpha\}$ is dense in H.

27. State and prove Hahn-Banach separation theorem.

28. Let X be a normed space and Y be a closed subspace of X. Show that X is a Banach space iff Y and $\frac{X}{Y}$ are Banach spaces in the induced norm and the quotient norm, respectively.

(2 × 4 = 8 weightage)