**16P302** (Pages : 2) Name: …………………

Reg. No: …………………

**THIRD SEMESTER M.Sc. DEGREE EXAMINATION, OCTOBER 2017**

(Regular/Supplementary/Improvement)

(CUCSS - PG)

**CC15P MT3 C12 - FUNCTIONAL ANALYSIS I**

(Mathematics)

(2015 Admission Onwards)

Time: Three hours Maximum: 36 Weightage

**Part A**

Answer **all** Questions. Each question carries 1 weightage

1. Prove or disprove: Property of completeness is shared by an equivalent metric.
2. State Minkowski's inequality for measurable functions.
3. For $1 \leq p < r < \infty$, prove that the normed space $l^p \subset l^r$.
4. Let $X$ be a normed space and $Y$ be a closed subspace of $X$. Prove that $X/Y$ is a normed space in the quotient norm.
5. State Riesz lemma.
6. Let $\langle\ ,\ \rangle$ be an inner product on a linear space $X$. Show that $|\langle x, y\rangle|^2 \leq \langle x, x\rangle\langle y, y\rangle$ for every $x, y \in X$.
7. Prove that $l^p$ with usual norm is not an inner product space unless $p = 2$.
8. Let $X$ be an inner product space and $E$ be an orthogonal subset of $X$ such that $0 \notin E$. Prove that $E$ is linearly independent.
9. Let $\{x_1, x_2, \dots, x_n\}$ be an orthogonal set in an inner product space $X$ and $k_1, k_2, \dots, k_n$ be scalars having absolute value 1. Show that $\|k_1x_1 + k_2x_2 + \dots + k_nx_n\| = \|x_1 + x_2 + \dots + x_n\|$.
10. Let $X$ be an inner product space and $Y$ be its subspace. Show that the best approximation to an element $x \in X$ from $Y$ is unique if it exists.
11. Let $X$ be a normed space over $\mathbb{K}$, $f \in X'$ and $f \neq 0$. Let $a \in X$ with $f(a) = 1$ and $r > 0$. Show that $U(a, r) \cap Z(f) = \emptyset$ iff $\|f\| \leq \frac{1}{r}$.
12. Prove that $c_{00}$ is not closed in $l^\infty$ as a normed space.
13. Let $X, Y$ be two normed spaces and $F : X \to Y$ be linear. Prove that $F$ is continuous iff $g \circ F$ is continuous for every $g \in Y'$.
14. Show that the dual $X'$ of every normed space $X$ is a Banach space.

(14×1=14 weightage)

## Part B

Answer any ***seven*** Questions. Each question carries 2 weightage

15. Show that the set of all polynomials in one variable is dense in $C([a,b])$ with the sup metric.
16. Let $E \subset \mathbb{R}$ be measurable. Show that the set of all simple measurable functions is dense in $L^{\infty}(E)$.
17. Show that every finite dimensional subspace of a normed space is complete.
18. Let $X$ be a normed space. Show that $X$ is finite dimensional iff the subset $\{x \in X : \|x\| \leq 1\}$ of $X$ is compact.
19. Let $X$ and $Y$ be normed spaces and $F : X \to Y$ be a linear map. Show that $F$ is a homeomorphism iff there exist $\alpha, \beta > 0$ such that $\beta\|x\| \leq \|F(x)\| \leq \alpha\|x\|$ for all $x \in X$.
20. State and prove Bessel's inequality.
21. Let $H = L^2([0,1])$, $x(t) = 0$, $t \in [0,1]$. Show that the best approximation to $x$ from the set $E = \left\{y \in L^2([0,1]) : \int_0^1 ty(t)\,dt = 1, \int_0^1 t^2 y(t)\,dt = 2\right\}$ is $y(t) = -72\,t + 100\,t^2, t \in [0,1]$.
22. Let $X$ be a normed space and $\{a_1, \dots, a_m\}$ be a linearly independent set in $X$. Show that there exist $f_1, \dots, f_n$ in $X'$ such that $f_j(a_i) = \delta_{ij}$, $1 \leq i, j \leq m$.
23. Prove that a Banach space cannot have a denumerable (Hamel) basis.
24. State and prove uniform boundedness principle.

**(7 × 2=14 weightage)**

## Part C

Answer **any two** Questions. Each question carries 4 weightage**.**

25. For $1 \leq p \leq \infty$, show that $l^p$ is complete.
26. Let $X$ and $Y$ be normed spaces and $F : X \to Y$ be a linear map such that the range $R(F)$ of $F$ is finite dimensional. Show that $F$ is continuous iff the zero space $Z(F)$ of $F$ is closed in $Y$. Can we drop the finite dimensionality of $R(F)$ ? Justify.
27. Let $H$ be non zero Hilbert space over $\mathbb{K}$. Prove that $H$ is separable iff $H$ has a countable orthonormal basis.
28. State and prove Hahn Banach Separation theorem.

**(2 × 4=8 weightage)**

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