17P307 (Pages: 2) Name: ……….…………………..

Reg. No…………………….

THIRD SEMESTER M.Sc. DEGREE EXAMINATION, NOVEMBER 2018

(CUCSS-PG)

Mathematics

CC17P MT3 C14 - FUNCTIONAL ANALYSIS

(2017 Admission)

Time: 3 Hours Maximum: 36 Weightage

Part A

Answer ***all*** questions. Each question carries 1 weightage.

1. If X is a separable metric space and $Y \subset X$, then show that Y is separable in the induced metric.
2. State Holder's and Minkowski's inequalities for the measurable functions.
3. Define operator norm of an operator on a normed space. Find the norm of the operator $\mathrm{F}: \mathrm{R}^2 \rightarrow \mathrm{R}^2$ defined by $\mathrm{F(x, y) = (y, 0)}$.
4. Let E_1 and E_2 be open subsets of a normed linear space X. Then Show that $E_1 + E_2$ is open in X.
5. State Korovkin's theorem and give a proper dense subset of $C[a, b]$.
6. Let X be a complex linear space and u be a real linear functional on it. Then give a complex linear functional f $on\ X$ such that $\mathrm{Re(f) = u}$.
7. Let X be a normed linear space over $K, f \in X^1, dual\ space\ of\ X, and\ f \neq 0$ Let $a \in X\ with\ f(a) = 1$ and $r > 0$. Then prove that $U(a, r) \cap Z(f) = \emptyset$ if and only if $\|f\| \leq \frac{1}{r}$
8. Let X be a normed space and $a \in X$ be a non zero vector. Then show that there exist some $f \in X^1$ with $(a) = \|a\|$.
9. Let Y be a subspace of a normed space X. Then show that $Y^0 \neq \emptyset$ if and only if $Y = X$, Where Y^0 is the interior of Y.
10. Let X be a normed space and $f: X \rightarrow K$ be linear. Then show that f is closed if and only if f is continuous.
11. State and prove bounded Inverse theorem.
12. Let X be a normed linear space which satisfies parallelogram law. Define an inner product on it which is compatible with the norm.
13. Let $\{x_1, x_2, \dots\dots x_n\}$ be an orthogonal set in an inner product space X and $k_1, k_2, \dots k_n$ be scalars with absolute value one. Show that

$$\|k_1x_1 + k_2x_2 + \ \dots\dots\ k_nx_n\| = \|x_1 + x_2 + \cdots . x_n\|$$

14. Give an Orthonormal basis for $L^2[-\pi,\pi]$

(14 x 1 = 14 Weightage)

Part B

Answer any ***seven*** questions. Each question carries 2 weightage.

15. Give an example of a linear discontinuous function between two normed spaces. Justify your answer.
16. Prove that a linear map F from a normed space X to a normed space Y is a homeomorphism if and only if $\exists\ \alpha, \beta > 0$ such that $\beta\|x\| \leq \|F(x)\| \leq \alpha\|x\|\ \forall\ x \in X$.
17. State true or false : A subset of a normed space is compact if and only if it is closed and bounded. Justify your answer.
18. Show that $\|.\|_1$, $\|.\|_2$ *and* $\|.\|_\infty$ are equivalent on K^n, Where K is either real or complex field.
19. Prove : A Banach space cannot have a denumerable basis.
20. Prove: Let X be a normed space and Y be a closed subspace of X .Then X is a Banach space if and only if Y and X/Y are Banach spaces in the induced norm and the quotient norm respectively.
21. Let X be a normed space E is a subset of X .Then E is bounded if and only if $f(E)$ is bounded in $K\ \forall f \in X^1$.
22. State and prove open mapping theorem.
23. Let X and Y be normed spaces and F : $X \to Y$ be linear. Then F is continuous if and only for every Cauchy sequence (x_n) in X , $F(x_n)$ is Cauchy in Y.
24. State and prove Bessel's inequality.

(7 x 2 = 14 Weightage)

Part C

Answer any ***two*** questions. Each question carries 4 weightage.

25. Let X be a normed space over a field K(either real or complex) and Y be a subspace of X. Then show that
    a) For $x \in X, y \in Y$ *and* $k \in K$, $\|kx + y\| \geq |k|\ dist(x, Y)$
    b) If Y is finite dimensional, then Y is complete.
26. State and prove Hahn –Banach extension theorem.
27. Let $X = \{x \in C[-\pi,\pi]; x(\pi) = x(-\pi)\}$ with sup norm. Then show that Fourier series of every x in a dense subset of X diverges at zero.
28. State and prove closed graph theorem.

(2 x 4 = 8 Weightage)

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