C 82511 (Pages : 3) Name..............................

Reg. No..............................

FOURTH SEMESTER M.Sc. DEGREE EXAMINATION, JUNE 2015

(CUCSS)

Mathemathics

MT 4C 15—FUNCTIONAL ANALYSIS—II

Time : Three Hours Maximum : 36 Weightage

Part A

Answer **all** *questions.*

Each question carries 1 weightage.

1. Let P be a bounded linear map from a normed space X into itself. If P is a projection, then prove that the range space $\mathcal{R}(\mathcal{P})$ and the null space $\mathcal{Z}(\mathcal{P})$ are closed in X.
2. Give an example of a closed linear map between Banach spaces which is not open.
3. Let X and Y be Banach spaces and F be a bounded linear, bijective map from X to Y. Using closed graph theorem, prove that F^{-1} is a bounded linear map from Y to X.
4. Let X be Banach space. Prove that the set of all bounded invertible operators on X is an open subset of the set of all bounded operators on X.
5. If A is an invertible bounded operator on a normed space X, then prove that $\sigma(A^{-1}) = \{k^{-1} : k \in \sigma(A)\}$ where $\sigma(A)$ is the spectrum of A.
6. If X is a finite dimensional normed space, then prove that its eigen spectrum, approximate eigen spectrum and the spectrum are the same.
7. Define reflexive normed spaces and give an example of it.
8. Let $E = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 = x_2\}$. Find $E^{\perp}$ and prove that it is a closed subspace of the Hilbert space $\mathbb{R}^2$.
9. Does projection theorem hold for incomplete inner product spaces ? Justify your answer.
10. Prove that in a finite dimensional Hilbert space weak convergent sequences are convergent.
11. Let H be a Hilbert space and let A be a bounded operator on H. Prove that $\|A\| = \|A^*\|$.
12. Let H be a Hilbert space and let A be a bounded operator on H. Prove that the closure of $\mathcal{R}(A^*)$ equals $\mathcal{Z}(A)^{\perp}$, where $\mathcal{R}(A^*)$ is the range space of A^* and $\mathcal{Z}(A)$ is the null space A.

Turn over

13. Let A be a normal operator on the Hilbert space H. If k is an eigen value of A, then prove th is an eigven value of A^* and the eigen vector of A corresponding to k is an eigen vector c corresponding to $\bar{k}$.

14. Give an example of a Hilbert-Schmidt operator on the Hilbert space l^2.

(14 × 1 = 14 weight

Part B

Answer any **seven** *questions.*
Each question carries 2 weightage.

15. Let X be a normed space over $\mathbb{C}$ and let $f : X \to \mathbb{C}$ be a linear map. Prove that f is closed if and if it is continuous.

16. Let X be a Banach space and let A be a bounded linear map on X. Prove that A is invertible i only if A is bounded below and the range of A is dense in X.

17. Let X be a Banach space over $\mathbb{C}$ and let A be a bounded operator on X. Prove that $\sigma(A)$ is a com subet of $\mathbb{C}$.

18. Let X and Y be normed spaces and let F be a bounded linear map from X to Y. Prove that transpose F' of F is a bounded linear map from Y to X and $\|F\| = \|F'\|$.

19. Prove that the dual X' of a reflexive normed space X is reflexive.

20. Let H be a Hilbert space and let F be a non-empty closed subspace of H. Prove that $H = F +$

21. Let H be a Hilbert space. Prove that the set of all normal operators on H is a closed subet of th of all bounded operators on H.

22. Let H be a Hilbert space and A be an operate on H. Prove that the spectrum of A is contain the closure of the numerical range $\omega(A)$.

23. Let A be a self adjoint operator on a finite dimensional Hilbert space H. Prove that every ro the characteristic polynomial of A is real.

24. Let A be a compact operator on a non-zero Hilbert space H. Prove that every non-zero approxi eigenvalue of A is an eigenvalue of A.

(7 × 2 = 14 weight

Part C

Answer any **two** *questions.*
Each question carries 4 weightage.

25. Let X be a non-zero Banach space over $\mathbb{C}$ and A be a bounded linear map on X. Prove tha spectrum of A is non-empty and its spectral radius r_σ is :

$$r_\sigma = \lim_{n\to\infty} \|A^n\|^{\frac{1}{n}}.$$

26. State and prove Riesz representation theorem for continuous linear functionals on a Hilbert space.

27. Let H be a Hilbert space over $\mathbb{C}$ and let A be a bounded linear map on H. Prove that :

    (i) k is a special value of A if and only if $\bar{k}$ is a spectral value of A^*.

    (ii) $\sigma_e(A) \subset \sigma_a(A)$ and $\sigma(A) = \sigma_a(A) \cup \{k : \bar{k} \in \sigma_e(A^*)\}$, where $\sigma(A)$, $\sigma_e(A)$ and $\sigma_a(A)$ are the spectrum, eigen spectrum and approximate eigen spectrum of A.

28. Let A be a non-zero compact self adjoint operator on a Hilbert space H over $\mathbb{C}$. Prove that there exist an infinite sequence $\{s_n\}$ of non-zero real numbers with $|s_1| \geq |s_2| \geq \ldots$ and an orthonormal set $\{u_1, u_2, \ldots\}$ in H such that

$$A(x) \sum_n s_n \langle x, u_n \rangle u_n.$$

(2 × 4 = 8 weightage)