16P402 (Pages: 2) Name…………………

Reg.No………………..

FOURTH SEMESTER M.Sc. DEGREE EXAMINATION, MARCH 2018

(Regular/Supplementary/Improvement)

(CUCSS - PG)

CC15 PMT4 C16 –DIFFERENTIAL GEOMETRY

(Mathematics)

(2015 Admission onwards)

Time: Three Hours Maximum:36 Weightage

Part A

Answer ***all*** questions. Each question carries ***1*** weightage.

1. Show that the graph of any function $f : \mathbb{R}^n \to \mathbb{R}$ is a level set for some function $F: \mathbb{R}^{n+1} \to \mathbb{R}$.
2. Check whether $f(x_1, x_2) = {x_1}^2 + {x_2}^2$ where $(x_1, x_2) \in \mathbb{R}^2$ is smooth and if smooth sketch the gradient vector field.
3. Show that if an n- surface S is represented both as $f^{-1}(c)$ and $g^{-1}(d)$ where $\nabla f(p) \neq 0$ and $\nabla g(p) \neq 0$ for all $p \in S$, $\nabla f(p) = \lambda \nabla g(p)$ for some real number $\lambda \neq 0$.
4. Show that the two orientations on the n −sphere${x_1}^2 + {x_2}^2 + \ldots\ldots\ldots + {x_{n+1}}^2 = r^2$ of radius $r > 0$ are given by $\mathbb{N}_1(p) = (p, \frac{p}{r})$and $\mathbb{N}_2(p) = (p, \frac{-p}{r})$.
5. Show that if$S$ is a connected n- surface in $\mathbb{R}^{n+1}$ and $g: S \to \mathbb{R}$ is smooth and takes only $+1$ and -1 then g is a constant.
6. Define Geodesic in an n-surface $S \subseteq \mathbb{R}^{n+1}$ and prove that geodesics have constant speed.
7. Prove that the velocity vector field along a parametrised curve α in an n-surface S is parallel if and only if α is a geodesic.
8. Let $\alpha(t) = \big(x(t), y(t)\big), t \in I$ be a local parametrisation of the oriented plane curve C, show that $\kappa o\ \alpha = \frac{x'y'' - y'x''}{[(x')^2 + (y')^2]^{3/2}}$ where κ curvature of C at p.
9. Prove that $\nabla_{\mathbb{v}}(\mathbb{X} + \mathbb{Y}) = \nabla_{\mathbb{v}}(\mathbb{X}) + \nabla_{\mathbb{v}}(\mathbb{Y})$ where $\mathbb{X}$ *and* $\mathbb{Y}$ are smooth vector fields on an open set Uin $\mathbb{R}^{n+1}$.
10. Find the length of the parametrised curve $\alpha: I \to \mathbb{R}^{n+1}$ given by $\alpha(t) = \big(\sqrt{2}\cos 2t, \sin 2t, \sin 2t\big)$, for $I = [0, 2\pi]$, n = 2
11. Let S be an oriented n- surface in $\mathbb{R}^{n+1}$and let $p \in S$. Define the first and second fundamental form of S at p.
12. Let S be an oriented n- surface in $\mathbb{R}^{n+1}$ and let$p \in S$. Give a formula for computing $\varkappa(p)$, the Gauss Kronecker curvature.
13. Show that a parametrised1- surface is simply a regular parametrised curve.

14. Show that if $S = f^{-1}(c)$is an n- surface in $\mathbb{R}^{n+k}$and $p \in S$, then the tangent space S_p to S at p is equal to the kernel of df_p.

(14 x 1 = 14 weightage)

Part B

Answer any ***seven*** questions. Each question carries ***2*** weightage

15. Find the Maximal Integral curve of the vector field $\mathbb{X}$ defined by $\mathbb{X}(x_1, x_2) = (x_2, -x_1)$through $p = (1, 1)$.
16. Let $U \subset \mathbb{R}^{n+1}$be open and $f: U \to \mathbb{R}$ be smooth. Let$p \in U$ be a regular point of f and let $c = f(p)$ then prove that the set of all vectors tangent to $f^{-1}(c)$ at p is equal to the orthogonal complement of $\nabla f(p)$.
17. State and prove Legrange Multiplier theorem.
18. Describe the spherical image of the n– surface,$-{x_1}^2 + {x_2}^2 + {x_3}^2 + \cdots \ldots \ldots \ldots + {x_{n+1}}^2$, $x_1 > 0$ when $n = 1$ and $n = 2$ oriented by$\frac{\nabla f}{\|\nabla f\|}$.
19. State and prove the existence of a maximal geodesic in an n- surface.
20. Prove that the Weingarten map $\mathcal{L}_p$is self adjoint.
21. The normal component of the acceleration is same for all parameterised curves α in S passing through p with velocity $\overline{\mathrm{v}}$.
22. Let C be a connected oriented plane curve and let $\beta : I \to C$ be a unit speed global parametrisation of C. Then prove that β is either one- one or periodic.
23. Find the Gaussian curvature of the ellipsoid $\frac{{x_1}^2}{4} + \frac{{x_2}^2}{4} + \frac{{x_3}^2}{9} = 1$
24. State and prove the Inverse Function Theorem for n – surfaces.

(7 x 2 = 14 weightage)

Part C

Answer any ***two*** questions. Each question carries ***4*** weightage.

25. Let S be a compact connected oriented n-surface in $\mathbb{R}^{n+1}$ expressed as the level set $f^{-1}(c)$of a smooth function $f: \mathbb{R}^{n+1} \to \mathbb{R}$with $\nabla f(p) \neq 0 \forall p \in S$, then prove that Gauss map is onto.
26. Let $\mathbb{X}$ be a smooth vector field on an open set $U \subseteq \mathbb{R}^{n+1}$ and let $p \in U$. Prove the existence and uniqueness of the maximal integral curve of S through p.
27. Let S be a compact connected oriented n-surface in $\mathbb{R}^{n+1}$. Prove that there exists a point $p \in S$ such that the second fundamental form at p is definite.
28. Obtain Gaussian curvature of an ellipsoid

(2 x 4 = 8 weightage)

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