17P401 (Pages: 2) Name……………………….

Reg. No…………………..

FOURTH SEMESTER M.Sc. DEGREE EXAMINATION, APRIL 2019

(CUCSS - PG)

(Mathematics)

CC15P MT4 C15 – FUNCTIONAL ANALYSIS II

(Improvement/Supplementary)

(2015, 2016 Admissions)

Time: Three Hours Maximum: 36 Weightage

Part A

Answer ***all*** questions. Each question carries 1 weightage.

1. State and prove bounded inverse theorem.
2. Let X be a normed space over $\mathbb{K}$ and $A, B \in BL(X)$. If $k \neq 0$, then prove that $k \in \sigma(AB)$ iff $k \in \sigma(BA)$
3. Let X be a normed space and $A \in BL(X)$. Prove that $k \in \sigma_a(A)$ iff there is a sequence $\{x_n\}$ in X such that $\|x_n\| = 1$ for each n and $\|A(x_n) - kx_n\| \to 0$ as $n \to \infty$
4. State True or False and justify. "Comparable norms preserve completeness".
5. State True or False and justify. "Strictly convex normed spaces are uniformly convex".
6. State True or False and justify. "Every continuous linear maps on a normed space is compact".
7. Let X be an inner product space and E is a subset of X. If F denotes the closure of span of E then prove that $F^{\perp} = E^{\perp}$
8. State Riesz representation theorem.
9. Let H be a Hilbert space and $A \in BL(H)$. Define $T: H' \to H$ by $T(f) = y_f$, where $f \in H'$and y_f is the representer of f. Show that $A^* = T\, A'\, T^{-1}$, where A' is the transpose of A
10. Prove that every bounded subset of a Hilbert space is weak bounded.
11. Let H be a Hilbert space. Prove that the set of all unitary operators is a closed subset of $BL(H)$
12. Let H be a Hilbert space and $A \in BL(H)$ be normal. Prove that the eigenvectors corresponding to distinct eigenvalues are orthogonal.
13. Let $H \neq \{0\}$ and $A \in BL(H)$. Prove that $\|A\| = \sup\{\sqrt{|k|} : k \in \sigma(A^*A)\}$
14. Let H be a Hilbert space and $A \in BL(H)$. Prove that A is compact iff A^* is compact.

(14 x 1 = 14 Weightage)

Part B

Answer any ***seven*** questions. Each question carries 2 weightage.

15. State and prove open mapping theorem.

16. Let X be a nonzero Banach space over $\mathbb{C}$ and $A \in BL(X)$. Prove that the spectrum $\sigma(A)$ of A is a non empty subset of $\mathbb{C}$
17. Let X be a normed space over $\mathbb{K}$. If its dual X' is reflexive then prove that X is reflexive.
18. Prove that every closed subspace of a reflexive normed space is reflexive.
19. Let X be a uniformly convex normed space and $\{x_n\}$ be a sequence in X such that $\|x_n\| \to 1$ and $\|x_n + x_m\| \to 2$ as $n, m \to \infty$. Then prove that $\{x_n\}$ is a Cauchy sequence.
20. State Projection theorem. Also prove that the completeness cannot be dropped from the assumption.
21. Let H be a Hilbert space and $A \in BL(H)$. Then prove that there exists a unique $B \in BL(H)$ such that $\langle A(x), y\rangle = \langle x, B(y)\rangle, x, y \in H$
22. Let H be a Hilbert space and $A \in BL(H)$ be self-adjoint. Prove that
$$\|A\| = \sup\{|\langle A(x), x\rangle| : x \in H, \|x\| \leq 1\}$$
23. Let $\mathbb{C}$ denotes the set of all complex numbers and let $H = \mathbb{C}^2$ over $\mathbb{C}$. Define $A \in BL(H)$ by $A(x) = (ax_1 + bx_2,\ cx_1 + dx_2)$, where $a, b, c, d \in \mathbb{C}$ are fixed and $x = (x_1, x_2)$. Show that A is a positive operator iff $a \geq 0$, $d \geq 0$, $c = b$ and $ad \geq |b|^2$
24. Let H be a Hilbert space and $A \in BL(H)$
Prove that $\sigma(A) = \sigma_a(A) \cup \{ k \in \mathbb{K} : \bar{k} \in \sigma_e(A^*)\}$

(7 x 2 = 14 Weightage)

Part C

Answer any ***two*** questions. Each question carries 4 weightage.

25. State and prove closed graph theorem.
26. Let X be a normed space and $A \in BL(X)$ be of finite rank. Prove that
$$\sigma_e(A) = \sigma_a(A) = \sigma(A)$$
27. Let H be a Hilbert space, G be a subspace of H and g be a continuous linear functional on G. Prove that there exists a unique continuous linear functional f on H such that $f = g$ on G and $\|f\| = \|g\|$
28. Let A be a nonzero compact self-adjoint operator on a Hilbert space H over $\mathbb{K}$. Prove that there exist a finite or infinite sequence $\{s_n\}$ of nonzero real numbers with $|s_1| \geq |s_2| \geq \cdots$ and an orthonormal set $\{u_1, u_2, \ldots\}$ in H such that
$A(x) = \sum_n s_n \langle x, u_n\rangle\, u_n, x \in H$. Also if the set $\{u_1, u_2, \ldots\}$ is infinite then prove that $s_n \to 0$ as $n \to \infty$.

(2 x 4 = 8 Weightage)

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