17P403 (Pages: 2) Name……….…………….

Reg. No…………………..

FOURTH SEMESTER M.Sc. DEGREE EXAMINATION, APRIL 2019

(CUCSS - PG)

(Mathematics)

CC15P MT4 E02 – ALGEBRAIC NUMBER THEORY

(Improvement/Supplementary)

(2015 & 2016 Admissions)

Time: Three Hours Maximum: 36 Weightage

Part A

Answer ***all*** questions. Each question carries 1 weightage.

1. Show that Q is not a cyclic group.
2. Find the order of the group G/H where G is free abelian with Z-basis, x, y, z and H is generated by 2x, 3y, 7z
3. Prove that an algebraic integer is a rational number iff it is a rational integer.
4. Let $\{\alpha_1, \alpha_2, --- , \alpha_n\}$ be any Q-basis of K. Then prove that

$$\Delta[\alpha_1, \alpha_2, --- , \alpha_n] = \det(T(\alpha_i\alpha_j)$$

5. Let K = Q(ξ) where $\xi = e^{\frac{2\pi i}{p}}$ for a rational prime p. In the ring of integers Z[ξ], show that $\alpha \in Z(\xi)$ is a unit iff $N_k(\alpha) = \pm 1$
6. Which of the following elements of Z[i] are irreducible? 1 + i, 5, 12i. Justify your answer.
7. Let D be an arbitrary domain, x be a non-zero non-unit element of D. Prove that x is irreducible iff <x> is maximal along the proper principal ideals of D.
8. Give an example of an integral domain which has no irreducible elements at all.
9. Let R be a ring and α be a maximal ideal of R. Show that R/α is a field.
10. Find all fractional ideals of Z[i]
11. Sketch the lattice in R^2 generated by (1, 1) and (2, 3) and find a fundamental domain for the lattice.
12. Show that the quotient group R/Z is isomorphic to the circle group S.
13. Let L be an m-dimensional lattice in R^n. Prove that R^n/L is isomorphic to $T^m \times R^{n-m}$
14. Let K = Q(θ) where $\theta^3 = 3$. What is the map $\sigma : K \to L^{st}$ in this case?

(14 x 1 = 14 Weightage)

Part B

Answer any ***seven*** questions. Each question carries 2 weightage.

15. Compute integral basis and discriminant for $Q(\sqrt{2}, \sqrt{3})$

16. Show that if K is a number field then K = Q(θ) for some algebraic number θ
17. Show that every number field K possesses an integral basis, and the additive group of the ring of integers is a free abelian group of rank n, where n is the degree of K
18. Show that the discriminant of Q(ξ), where $\xi = e^{\frac{2\pi i}{p}}$ and p is an odd prime is $(-1)^{\frac{p-1}{2}} p^{p-2}$
19. Let K be a number field of degree n. Prove that D, the ring of integers of K, is a free abelian group of rank n
20. Show that every principal ideal domain is a unique factorization domain.
21. Show that if a, b are non-zero ideals of the ring of integers D of a number field K, then there exists α∈a such that αa^{-1} + b = D
22. Prove that an integral domain D is noetherian iff D satisfies the maximal condition.
23. If x, y, z are integers such that $x^2 + y^2 = z^2$, prove that at least one of x, y, z is a multiple of 3
24. Let K = Q(ξ), where $\xi = e^{\frac{2\pi i}{p}}$ for an odd prime p. Show that the only roots of unity in K are $\pm\xi^s$ for integers s

(7 x 2 = 14 Weightage)

Part C

Answer any ***two*** questions. Each question carries 4 weightage.

25. (a) Show that the algebraic integers form a subring of the field of algebraic numbers.
    (b) Let K = Q(θ) be a number field. Prove that if all k-conjugates of θ are real, then the discriminant of any basis is positive.
26. Show that in a domain in which factorization into irreducibles is possible, factorization is unique if every irreducible is prime.
27. Factorize the ideal <18> in $Z[\sqrt{-17}]$
28. Sketch a proof of Kummer's theorem, including a proofs of some of the main steps.

(2 x 4 = 8 Weightage)

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