17P402 (Pages: 3) Name…………………..…….

Reg.No………………..……..

FOURTH SEMESTER M.Sc. DEGREE EXAMINATION, APRIL 2019

(CUCSS - PG)

(Mathematics)

CC15P MT4 C16 / CC17P MT4 E14 – DIFFERENTIAL GEOMETRY

(Regular/Improvement/Supplementary)

(2015 Admission onwards)

Time: Three Hours Maximum: 36 Weightage

Part A

Answer ***all*** questions. Each question carries 1 weightage.

1. Find the integral curve through the point $p = (1, 1)$ of the vector field $\mathbb{X}(p) = (p, X(p))$, where $X(p) = (0,1)$ on $\mathbb{R}^2$
2. Show that the graph of any function $f: \mathbb{R}^n \to \mathbb{R}$, is a level set for some function $F: \mathbb{R}^{n+1} \to \mathbb{R}$
3. Let $f: U \to \mathbb{R}$ be a smooth function, where $U \subset \mathbb{R}^{n+1}$ is an open set, and let $\alpha: I \to U$ be a parametrized curve. Show that $(f \circ \alpha)$ is constant if and only if $\dot{\alpha}(t) \perp \nabla f(\alpha(t))$ for all $t \in I$
4. Sketch the cylinder $f^{-1}(0)$ where $f(x_1, x_2, x_3) = x_1 - {x_2}^2$
5. Let S be an oriented n-surface in $\mathbb{R}^{n+1}$, with orientation $\mathbb{N}$ and let $p \in S$. Show that an ordered basis for S_p is inconsistent with $\mathbb{N}$ if and only if it is consistent with $-\mathbb{N}$
6. Describe the spherical image of the 2-surface $x_2^2 + x_3^2 = 1$ oriented by $\frac{\nabla f}{\|\nabla f\|}$, where $f(x_1, x_2, x_3) = x_2^2 + x_3^2$
7. Find the velocity, the acceleration and the speed of the parameterized curve
$$\alpha(t) = (\cos t, \sin t, t)$$
8. Define Euclidean parallel and Levi-Civita parallel.
9. Compute $\nabla_v f$ where $f: \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x_1, x_2) = 2{x_1}^2 + 3{x_2}^2$, $v = (1, 0, 2, 1)$
10. Let $g: I \to R$ be a smooth function and let C denote the graph of g. Show that the curvature of C at the point $(t, g(t))$ is $\frac{g''(t)}{(1+(g'(t))^2)^{3/2}}$, for an appropriate choice of orientation.
11. Find the length of the parameterized curve $\alpha: I \to \mathbb{R}^3$ where $I = [-1,1]$ and
$$\alpha(t) = (\cos 3t, \sin 3t, 4t)$$
12. Define normal section of an n-surface.
13. Define Weingarten map for parametrized n-surfaces.
14. State inverse function theorem for n-surfaces.

$(\mathbf{14 \times 1 = 14\ Weightage})$

 Turn Over

Part B

Answer any ***seven*** questions. Each question carries 2 weightage.

15. Define level set of a function $f: U \to \mathbb{R}$, $U \subset \mathbb{R}^{n+1}$. Sketch the level sets $f^{-1}(c)$ for n = 0 and 1 for the function $f(x_1, \dots, x_{n+1}) = x_{n+1}; C = -1,\ 0,\ 1,\ 2$
16. Define a vector field on $\mathbb{R}^{n+1}$. Sketch the vector field $\mathbb{X}(p) = (p,\ X(p))$ on $\mathbb{R}^2$ where
$$X(x_1, \quad x_2) = (-x_1, -x_2)$$
17. Let S be unit circle $x_1{}^2 + x_2{}^2 = 1$ and define $g: \mathbb{R}^2 \to \mathbb{R}$ by $g(x_1,\ x_2) = ax_1{}^2 + 2bx_1x_2 + c\,x_2{}^2$ where $a,\ b,\ c \in \mathbb{R}$. Show that the extreme point of g on S are the eigenvector of a matrix $\begin{bmatrix} a & b \\ b & c \end{bmatrix}$
18. Show that the two orientation on the n-sphere $x_1{}^2 + x_2{}^2 + \cdots . + x_{n+1}{}^2 = r^2$ of radius $r > 0$ are given by $\mathbb{N}_1(p) = (p, \frac{p}{r})$ & $\mathbb{N}_2(p) = (p, \frac{-p}{r})$
19. Prove that in an n-plane, parallel transport is path independent.
20. Show that the Weingarten map L_p is self adjoint.
21. Prove that for each 1-form ω on open set U in $\mathbb{R}^{n+1}$, there exist unique functions $f_i: U \to R$, $i = 1, 2, \dots, n+1$, such that $\omega = \sum_{i=1}^{n+1} f_i - dx_i$. Show further that ω is smooth if and only if each f_i is smooth.
22. Let S be an oriented n-surface in $\mathbb{R}^{n+1}$ and let $\mathbb{v}$ be a unit vector in $S_p, p \in S$. Prove that there exists an open set $\mathbb{V} \subset \mathbb{R}^{n+1}$ containing p such that $S \cap \mathcal{N}(\mathbb{v}) \cap \mathbb{V}$ is a plane curve. Also prove that the curvature at p of this curve (suitably oriented) equals the normal curvature$K(\mathbb{v})$
23. Let $\varphi: U \to \mathbb{R}^3$ be given by $\varphi(\theta, \phi) = (r\cos\theta\sin\phi,\ r\sin\theta\sin\phi,\ r\cos\phi)$ where $U = \{(\theta, \phi) \in\ \mathbb{R}^2 : 0 < \phi < \pi\}$ and $r > 0$.Then show that φ is a parametrized 2-surface.
24. Show for a parameterized n-surface $\varphi: U \to \mathbb{R}^{n+1}$ in $\mathbb{R}^{n+1}$ and for $p \in U$, there exists an open set $U_1 \subset U$ about p such that $\varphi(U_1)$ is an n-surface in $\mathbb{R}^{n+1}$

($7 \times 2 = 14$ Weightage)

Part C

Answer any ***two*** questions. Each question carries 4 weightage.

25. What do you mean by a vector at a point p tangent to a level set in $f^{-1}(c)$ of a smooth function $f: U \to \mathbb{R}$, where U is open in $\mathbb{R}^{n+1}$? Show that, for such a function f with a regular point $p \in U$, the set of all vectors tangent to $f^{-1}(c)$ at p is equal to $[\nabla f(p)]^{\perp}$

26. Let S be an n-surface in $\mathbb{R}^{n+1}$, $p \in S$ and $\mathbb{v} \in S_p$. Then prove there exists an open interval I containing 0 and a geodesic $\alpha: I \to S$ such that:
    (i) $\alpha(0) = p$ and $\dot{\alpha}(0) = \mathbb{v}$
    (ii) $\beta: \hat{I} \to S$ is any other geodesic in S with $\beta(0) = p$ and $\dot{\beta}(0) = \mathbb{v}$ then $\hat{I} \subset I$ and $\alpha(0) = \beta(0)$ for all $t \in \hat{I}$
27. Let $\alpha: I \to \mathbb{R}^3$ be a unit speed parametrized curve in $\mathbb{R}^3$ such that $\dot{\alpha}(t) \times \ddot{\alpha}(t) \neq 0$ for all $t \in I$. Let $\mathbb{T}, \mathbb{N}$ and $\mathbb{B}$ denote the vector fields along α defined by $\mathbb{T}(t) = \dot{\alpha}(t)$, $\mathbb{N}(t) = \ddot{\alpha}(t) / \|\ddot{\alpha}(t)\|$ and $\mathbb{B}(t) = \mathbb{T}(t) \times \mathbb{N}(t)$ for all $t \in I$
    (i) Show that $\{\mathbb{T}(t), \mathbb{N}(t), \mathbb{B}(t)\}$ is orthonormal for all $t \in I$
    (ii) Show that there exist a smooth function $\kappa: I \to R$ and $\tau: I \to S$ such that $\dot{\mathbb{T}} = \kappa\mathbb{N}$, $\dot{\mathbb{N}} = -\kappa\mathbb{T} + \tau\mathbb{B}$, $\dot{\mathbb{B}} = -\tau\mathbb{N}$
28. (i) Let S be a compact oriented n-surface in $\mathbb{R}^{n+1}$. Prove that there exists a point p on S such that the second fundamental form of S at p is definite.
    (ii) Find the Gaussian curvature $K: S \to \mathbb{R}$, where S is the ellipsoid $\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} + \frac{x_3^2}{c^2} = 1$, ($a, b$ and c all $\neq 0$)

($2 \times 4 = 8$ Weightage)

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