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Name..................................

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FIRST SEMESTER M.Sc. DEGREE EXAMINATION, DECEMBER 2014

(CUCSS)

Mathematics

MT 1C 03—REAL ANALYSIS—I

Time : Three Hours

Maximum : 36 Weightage

Part A (Short Answer Questions)

Answer **all** *questions.*
Each question has 1 weightage.

1. Construct a compact set of real numbers whose limit points form a countable set.
2. Define perfect set. Give an example of a perfect set which is not bounded.
3. Prove that the set of all interior points of a set E is open.
4. Prove that a uniformly continuous function of a uniformly continuous functions is uniformly continuous.
5. Is inverse of a bijective continuous function continuous ? Justify your answer.
6. Identify the type of discontinuity of the following function :

$$f(x) = \begin{cases} \sin \dfrac{1}{x} & (x \neq 0) \\ 0 & (x = 0) \end{cases}$$

at $x = 0$.

7. State Taylors theorem.
8. Evaluate $\lim\limits_{x \to 0} \dfrac{\sin x}{x}$.
9. Is mean value theorem real valued functions valid for vector valued functions ? Justify your answer.
10. Let f be a bounded real valued function defined on $[a, b]$ and $|f|$ be Riemann integrable on $[a, b]$. Is f Riemann integrable ? Justify your answer.
11. Let f be a bounded function and α be a monotonic increasing function on $[a, b]$. If the partition P′ is a refinement of the partition P of $[a, b]$, then prove that $\mathrm{U}(\mathrm{P}', f, \alpha) \leq \mathrm{U}(\mathrm{P}, f, \alpha)$.

Turn over

12. Let γ be defined on $[0, 2\pi]$ by $\gamma(t) = e^{i2t}$. Prove that γ is rectifiable.

13. Define uniform convergence.

14. Prove that every function in an equicontinuous family of functions is continuous.

(14 × 1 = 14 weighta

Part B

Answer any **seven** *from the following* **ten** *questions.*
Each question has weightage 2.

15. Prove that finite intersection of open sets is open. Is it true in the case of arbitrary intersecti Justify your answer.

16. Prove that infinite subset of a countable set is countable.

17. For $x, y \in \mathbb{R}^1$, let $d(x, y) = \max\{|x|, |y|\}$. Prove that d is a metric. Which subsets of the resul metric space are open ?

18. Let f be a continuous mapping of a metric space X into a metric space Y and let E be a d subset of X. Prove that $f(E)$ is a dense subset of $f(X)$.

19. Let f be a real valued uniformly continuous function on the bounded set E in $\mathbb{R}^1$. Prove tha bounded on E.

20. Let f be a real valued differential function on (a, b). If $f'(x) = 0$ for all $x \in (a,b)$, then prove f is a constant.

21. Let f be a bounded function and α be a monotonic increasing function on $[a, b]$. Prove that Riemann-Steiltjes integrable with respect to α on $[a, b]$, then $|f|$ is Riemann-Steiltjes integ with respect to α on $[a, b]$ and

$$\left|\int_a^b d\, d\alpha\right| \le \int_a^b |f|\, d\alpha.$$

22. Let f be Riemann integrable on $[a, b]$ and let F be a differentiable function on $[a, b]$ such $F' = f$. Prove that $\int_a^b f(x)\, dx = F(b) - F(a)$.

23. Let $\{f_n\}$ be a sequence of functions defined on E such that $|f_n(x)| \le M_n$ for all $n = 1, 2, \ldots$ $x \in E$. Prove that $\sum_n f_n$ converges uniformly on E if $\sum_n M_n$ converges.

24. For $n = 1, 2 \ldots$ and x real let $f_n(x) = \frac{x}{1 + nx^2}$. Show that $\{f_n\}$ converges uniformly.

(7 × 2 = 14 weightage)

Part C

Answer any **two** *from the following* **four** *questions.*
Each question has weightage 4.

25. (a) Prove that a finite set has no limit points.

(b) Let P be a non-empty perfect set in $\mathbb{R}^k$. Prove that P is uncountable.

26. (a) Prove that compact subsets of a metric space are closed.

(b) Let E be a subset of the real line $\mathbb{R}^1$. Prove that E is connected if and only if it satisfies the following property : If $x \in E$, $y \in E$ and $x < z < y$, then $z \in E$.

27. (a) Let f be defined on $[a, b]$. If f has a local maximum at a point x and if $f'(x)$ exists, then prove that $f'(x) = 0$.

(b) Let f be a continuous function and α be monotonic increasing function on $[a, b]$. Prove that f is Riemann-Steiltjes integrable with respect to α on $[a, b]$.

28. If $\{f_n\}$ be a sequence of functions on E and if $f_n \to f$ uniformly on E, then prove that f is continuous on E.

(2 × 4 = 8 weightage)