**16P101** (Pages:3) Name: ……………………….

Reg. No……………………...

# FIRST SEMESTER M.Sc. DEGREE EXAMINATION, NOVEMBER 2016

(Regular/Supplementary/Improvement)

(CUCSS-PG)

## CC15P MT1 C01 – ALGEBRA-I

(Mathematics)

(2015 Admission Onwards)

Time: Three Hours Maximum: 36 Weightage

### Part A

Answer **all** Questions

Each question carries 1 weightage

1. Define an isometry of $\mathbb{R}^2$ and give an example for it.
2. Find the order of the element $(3,10,9)$ in $\mathbb{Z}_4 \times \mathbb{Z}_{12} \times \mathbb{Z}_{15}$ .
3. Show that factor group of a cyclic group is cyclic.
4. Show that for two binary words of same length $d\,(u,v) = wt\,(u-v)$.
5. Show that $\mathbb{Z}$ has no composition series.
6. Let $X$ be a $G$-set. For $x_1, x_2 \in X$, let $x_1 \sim x_2$ if and only if there exists $g \in G$ such that $g\,x_1 = x_2$ . Show that $\sim$ is an equivalence relation on $X$.
7. Let $G$ be a group of order $p^n$ and let $X$ be a $G$-set. Prove that $|X| \equiv |X_G| (\text{mod p})$ where $p$ is a prime number.
8. Find the conjugate classes of $S_3$ and write the class equation.
9. Find the reduced form and the inverse of the reduced form of $a^2b^{-1}b^3a^3c^{-1}c^4b^{-2}$.
10. Consider the evaluation homomorphism $\varphi_5 : \mathbb{Q}[x] \to \mathbb{R}$. Find five elements in the kernal of $\varphi_5$.
11. If $F$ is a field and $a \neq 0$, is a zero of $f(x) = a_0 + a_1x + \cdots + a_nx^n$ in $F[X]$. Show that $\frac{1}{a}$ is a zero of $a_n + a_{n-1}x + \cdots + a_0x^n$.
12. Write the element $(1+3j)(4+2j-k)$ of $Q$ in the form $a_1 + a_2i + a_3j + a_4k$ for $a_i \in \mathbb{R}$.
13. Let $N$ be an ideal of a ring $R$. Then prove that $\Upsilon : R \to R/N$ given by $\Upsilon(x) = x + N$ is a ring homomorphism with kernal $N$.
14. Give an example to show that factor ring of an integral domain may be a field.

(14× 1 = 14 Weightage)

## Part B

Answer **any 7** Questions.

Each question carries 2 weightage

15. Prove that converse of Lagrange's theorem is not true.
16. Find isomorphic refinements of the two series $\{0\} < \langle 18 \rangle < \langle 3 \rangle < \mathbb{Z}_{72}$ and $\{0\} < \langle 24 \rangle < \langle 12 \rangle < \mathbb{Z}_{72}$.
17. Show that the group $\mathbb{Z}_m \times \mathbb{Z}_n$ is isomorphic to $\mathbb{Z}_{mn}$ if and only if $m$ and $n$ are relatively prime.
18. Prove that if $H$ is a subgroup of a finite group $G$ , then $(N[H] : H) \equiv (G : H)(mod\ p)$.
19. Find the number of orbits in $\{1,2,3,4,5,6,7,8\}$ under the cyclic group $< (1\ 3\ 5\ 6) >$ of $S_8$.
20. Prove that for a prime number $p$, every group $G$ of order $p^2$ is abelian.
21. Show that every group of order $(35)^3$ has a normal subgroup of order $125$.
22. Show that the equation $x^2 = 2$ has no solution in rational numbers.
23. Find $q(x)$ and $r(x)$ where $f(x) = x^5 - 2x^4 + 3x - 5$ and $g(x) = 2x + 1$ in $\mathbb{Z}_{11}[x]$.
24. Prove that if $F$ is a field, then every non constant polynomial $f(x) \in F[x]$ can be factored in $F[x]$ into a product of irreducible polynomials, the irreducible polynomials being unique except for order and for unit factors in $F$.

(7× 2 = 14 Weightage)

## Part C

Answer **any 2** Questions.

Each question carries 4 weightage.

25. Show that the set of all commutators $aba^{-1}b^{-1}$ of a group $G$ generates a normal subgroup $C$ of $G$ and $G/C$ is abelian. Furthermore $G/N$ is abelian if and only if $C \leq N$. Find the centre $Z(D_4)$ and the commutator subgroup $C$ of the group $D_4$.
26. State and prove second isomorphism theorem. Let $\varphi: \mathbb{Z}_{12} \rightarrow \mathbb{Z}_3$ be a homomorphism such that $\varphi(1) = 2$.
    a) Find the Kernel $K$ of $\varphi$.
    b) List the cosets in $\mathbb{Z}_{12}/K$ showing the elements in each coset.
    c) Give the correspondence between $\mathbb{Z}_{12}/K$ and $\mathbb{Z}_3$ given by map $\psi$ described in first isomorphism theorem.
27. Let $P_1$ and $P_2$ be Sylow $p$-subgroups of a finite group $G$. Prove that $P_1$ and $P_2$ are conjugate subgroups of $G$. Let $G$ be a finite group and let $p$ divides $|G|$. Prove that if $G$ has only one proper Sylow $p$-subgroup, it is a normal subgroup, so $G$ is not simple.

28. State and prove Eisenstein's theorem. Using Eisenstein's theorem, prove that the cyclotomic polynomial $\varphi_p(x) = \frac{(x^p-1)}{(x-1)} = x^{p-1} + x^{p-2} + \cdots + x + 1$ is irreducible over $Q$ for any prime $p$.

(2× 4 = 8 Weightage)

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