19P102A (Pages: 3) Name: …………………….

Reg. No…………………..

FIRST SEMESTER M.Sc. DEGREE EXAMINATION, NOVEMBER 2019

(Supplementary/Improvement)

(CUCSS-PG)

CC15P MT1 C02/CC17P MT1 C02 – LINEAR ALGEBRA

(Mathematics)

(2015 to 2018 Admissions)

Time: Three Hours Maximum: 36 Weightage

Part A

Answer ***all*** questions. Each question carries 1 weightage

1. Prove that the only subspaces of $\mathbb{R}^1$ are $\mathbb{R}^1$ and the zero subspace.
2. Prove that any subset of a linearly independent set is linearly independent.
3. Let W be the set of matrices of the form $\begin{bmatrix} x & -x \\ y & z \end{bmatrix}$, where x, y, z are elements of a field F. Find $\dim W$. Justify your answer.
4. Find the coordinates of the vector $(2,3)$ of $\mathbb{R}^2$ with respect to the basis $\mathfrak{B} = \{(1,1),(1,2)\}$.
5. Find two linear operators T and U on $\mathbb{R}^2$ such that $TU = 0$ but $UT \neq 0$.
6. Describe explicitly an isomorphism from the space of complex numbers over the Real field onto the space $\mathbb{R}^2$.
7. Let V and W be vector spaces over the field F, and let T be a linear transformation from V into W. Prove that the null space of T^t is the annihilator of the range of T.
8. If f is a non-zero linear functional on a finite dimensional vector space V over a field F, then prove that the null space N_f is a hyper space of V.
9. Let F be a field and let f be the linear functional on F^2 defined by $f(x,y) = ax + by$. For the linear operator $T(x,\ y) = (-y,\ x)$ and let $g = T^t f$. Find $g(x,y)$.
10. Find a 3×3 matrix for which the minimal polynomial is x^2.
11. Let T be the linear operator on $\mathbb{R}^2$, the matrix of which in the standard ordered basis is $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Prove that the only subspaces of $\mathbb{R}^2$ invariant under T are $\mathbb{R}^2$ and the zero space.
12. Prove that any projection is diagonalizable.
13. State and prove Cauchy-Schwarz inequality in an inner product space.

14. Give $\mathbb{R}^3$ with the standard inner product. Find the orthogonal projection of the vector $(1,2,3)$ on the subspace W that is spanned by the vector $(3,2,1)$.

(14 x 1 = 14 Weightage)

Part B

Answer any ***seven*** questions. Each question carries 2 weightage.

15. Let V be an n-dimensional vector space over the field F, and let $\mathfrak{B}$ and $\mathfrak{B}'$ be two ordered bases of V. Then prove that there is a unique, necessarily invertible, $n \times n$ matrix P with entries in F such that (i) $[\alpha]_{\mathfrak{B}} = P[\alpha]_{\mathfrak{B}'}$ (ii) $[\alpha]_{\mathfrak{B}'} = P^{-1}[\alpha]_{\mathfrak{B}}$ for every vector α in V.
16. Let V be the vector space of all functions from $\mathbb{R}$ into $\mathbb{R}$; let V_e be the subset of even functions, $f(-x) = f(x)$; let V_0 be the subset of odd functions $f(-x) = -f(x)$. Prove that
(i) V_e and V_0 are subspaces of V
(ii) $V_e \oplus V_0 = V$.
17. Let T be a linear transformation from V into W. Prove that T is non-singular if and only if T carries each linearly independent subset of V onto a linearly independent subset of W.
18. Let V be a finite-dimensional vector space over the field F, and let $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ be a basis for V. Prove that there is a unique dual basis $\{f_1, f_2, \dots, f_n\}$ for the dual space such that $f_i(\alpha_j) = \delta_{ij}$ and for each linear functional f on V we have $f = \sum_{i=1}^{n} f(\alpha_i) f_i$ and for each vector α in V we have $\alpha = \sum_{i=1}^{n} f_i(\alpha)\alpha_i$.
19. Prove that the double dual space of a vector space V is isomorphic to the space itself.
20. Let A is a $m \times n$ matrix over the field F. Prove that the row rank of A is equal to the column rank of A.
21. Let T be the linear operator on $\mathbb{R}^3$ which is represented in the standard ordered basis by the matrix $\begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix}$. Find the characteristic and minimal polynomials of T.
22. Let T be a linear operator on V and let U be any operator on V which commutes with T. Prove that the range and null space of U are invariant under T.
23. Find a projection E which projects $\mathbb{R}^2$ onto the subspace spanned by $(1,-1)$ along the subspace spanned by $(1,2)$.
24. Let W be a subspace of an inner product space V and let β be a vector in V. Prove that the vector α in W is a best approximation to β by vectors in W if and only if $\beta - \alpha$ is orthogonal to every vectors in W.

(7 x 2 = 14 Weightage)

Part C

Answer any ***two*** questions. Each question carries 4 weightage.

25. (a) Prove that in a finite dimensional vector space V every non-empty linearly independent set of vectors is part of a basis.
(b) Find three vectors in $\mathbb{R}^3$ which are linearly dependent, and are such that any two of them are linearly independent.
26. (a) Let V be an n-dimensional vector space over the field F, and let W be an m-dimensional vector space over F. Prove that the space of linear transformation $L(V,W)$ is finite dimensional and has dimension mn.
(b) Let F be a field and let T be the linear operator on F^2 defined by $T(x,y) = (x+y, x)$. Prove that T is invertible and find T^{-1}.
27. (a) Let V be a finite dimensional vector space over the field F and let T be a linear operator on V. Prove that T is diagonalizable if and only if the minimal polynomial for T has distinct roots.
(b) What is the minimal polynomial for the identity operator on V? What is the minimal polynomial for the zero operator on V?
28. Let W be a finite dimensional subspace of an inner product space V and let E be the orthogonal projection of V on W. Prove that
(a) E is a linear transformation of V onto W.
(b) E is an idempotent.
(c) $W^{\perp}$ is the null space of E.
(d) $V = W \oplus W^{\perp}$
(e) $I - E$ is the orthogonal projection of V on $W^{\perp}$.
(f) $I - E$ is an idempotent linear transformation of V onto $W^{\perp}$ with null space W.

(2 x 4 = 8 Weightage)

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