19P102 (Pages: 2) Name: ………………….

Reg. No………………..

FIRST SEMESTER M.Sc. DEGREE EXAMINATION, NOVEMBER 2019

(CUCSS PG)

CC19P MTH1 C02 – LINEAR ALGEBRA

(Mathematics)

(2019 Admission Regular)

Time: Three Hours Maximum: 30 Weightage

PART A (Short Answer questions)

Answer ***all*** questions. Each question carries 1 weightage.

1. Is the vector (3, -1, 0, -1) in the subspace of R^4 spanned by the vectors $(2,-1,3,2)$, $(-1,1,2,-3)$ and $(-1,1,9,-5)$? Justify.
2. Show that the vectors $(1,1,0,0)$, $(0,0,1,1)$, $(1,0,0,4)$ and $(0,0,0,2)$ form a basis for R^4.
3. If T is a linear operator on C^3 for which $\mathrm{T}\varepsilon_1 = (1,0,\ \mathrm{i})$, $\mathrm{T}\varepsilon_2 = (0,1,1), \mathrm{T}\varepsilon_3 = (\mathrm{i},1,0)$. Is T invertible. Give reason.
4. T is a linear operator on C^2 defined by $\mathrm{T}(\mathrm{x}_1,\ \mathrm{x}_2) = (\mathrm{x}_1,\ 0)$. Let $\mathrm{B}' = \{(1,\ i),\ (-i,\ 2)\}$ be an ordered basis. What is the matrix of T in this ordered basis B'?.
5. Prove that similar matrices have the same characteristic polynomials. .
6. Let F be a field and f be the linear functional on F^2 defined by $f(x,y) = 3x - 2y$. Write an expression for $(T^t f)(x,\ y)$ if $T(x,\ y) = (x-y,\ 2x)$.
7. Find out the characteristic values of an $n \times n$ triangular matrix.
8. Let R be the range of projection E then $\beta \in R$ if and only if $E\beta = \beta$

(8 × 1 = 8 Weightage)

PART B

Answer any ***two*** questions from each unit. Each question carries 2 weightage.

UNIT I

9. Let V be the space of all polynomial functions from R into R which have degree less than or equal to 2. Let t be a fixed real number, define $g_i(x) = (x+t)^{i-1}$, $i = 1,2,3$. Prove that $B = \{g_1, g_2, g_3\}$ is a basis for V. If $\mathrm{f(x)} = \mathrm{c}_0 + \mathrm{c}_1\mathrm{x} + \mathrm{c}_2\mathrm{x}^2$, what are the coordinates of f in the ordered basis B?
10. Let V and W be vector spaces over the field F and let T be a linear transformation from V into W. If V is finite dimensional, prove that $rank\ (T) + nullity\ (T) = dim\ V$
11. Show that every n-dimensional vector space over the field is isomorphic to the space F^n

UNIT II

12. Let V be finite dimensional vector space and let $B = \{\alpha_1, \dots, \alpha_n\}$ and $B' = \{\alpha_1', \dots, \alpha_n'\}$ be ordered bases for V. Suppose T is a linear operator on V. If $P = [P_1, \dots, P_n]$ is the $n \times n$ matrix with columns $P_j = [\alpha_j']_B$ then show that $[T]_{B'} = P^{-1}[T]_B P$

13. Let V be a finite dimensional vector space over the field F. For each vector α in V, define $L_\alpha(f) = f(\alpha),\ f \in V^*$. Prove that the mapping $\alpha \to L_\alpha$ is an isomorphism of V onto V^{**}
14. Let V be a finite dimensional vector space over the field F and T be a linear operator on V. Prove that T is diagonalizable if and only if the minimal polynomial for T has the form $p = (x - c_1) \ldots\ldots\ldots (x - c_k)$ where $c_1, \ldots, c_k$ are distinct elements of F

UNIT III

15. Let V be an inner product space W a finite subspace of V and E is the orthogonal projection of V on W then prove that the mapping $\beta \to \beta - E_\beta$ is an orthogonal projection of V on $W^\perp$
16. If $V = W_1 \oplus \ldots\ldots \oplus W_k$, then prove that there exist k linear operators $E_1, \ldots\ldots, E_k$ on V such that
    - i) each E_i is a projection
    - ii) $E_iE_j = 0$, if $i \neq j$
    - iii) $I = E_1 + \ldots\ldots + E_k$
    - iv) The range of E_i is W_i
17. State and prove Bessel's inequality.

(6 x 2 = 12 Weightage)

PART C

Answer any ***two*** questions. Each question carries 5 weightage.

18. Let V and W be finite dimensional vector spaces over F such that $\dim V = \dim W$. If T is a linear transformation from V into W, prove that the following are equivalent:
    (i) T is invertible (ii) T is non-singular (iii) T is onto
    (iv) If $\{\alpha_1, \ldots, \alpha_n\}$ is a basis for V then $\{T\alpha_1, \ldots, T\alpha_n\}$ is a basis for W
    (v) There is some basis $\{\alpha_1, \ldots, \alpha_n\}$ for V such that $\{T\alpha_1, \ldots, T\alpha_n\}$ is a basis for W
19. (a) If S is any subset of V, prove that $(S^0)^0$ is the subspace spanned by S
    (b) Let T be a linear operator on the finite dimensional space V. Let $c_1, \ldots\ldots, c_k$ be the distinct characteristic values of T and let W_i be the characteristic space associated with the value c_i. If $W = W_1 + \ldots\ldots + W_k$, prove that $\dim W = \dim W_1 + \ldots\ldots\ldots + \dim W_k$.
20. Let T be a linear operator on an n-dimensional vector space V. Prove that the characteristic and minimal polynomials for T have the same roots, except for multiplicities. Find the minimal polynomial for T represented in the standard ordered basis by the matrix
$$\begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix}$$
21. (a) Explain Gram-Schmidt Orthogonalization process
    (b) Consider the vectors $\beta_1 = (3, 0, 4), \beta_2 = (-1, 0, 7), \beta_3 = (2, 9, 11)$ in R^3 with standard inner product. Find an orthogonal basis for R^3

(2 x 5 = 10 Weightage)

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